56. Muon Anomalous Magnetic Moment

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The Dirac equation predicts a muon magnetic moment, $\vec{M} = g_\mu \frac{e}{2m_\mu}\vec{S}$, with gyromagnetic ratio $g_\mu = 2$. Quantum loop effects lead to a small calculable deviation from $g_\mu = 2$, parameterized by the magnetic anomaly[1]

$$a_\mu \equiv \frac{g_\mu - 2}{2} \ . \tag{56.1}$$

That quantity can be accurately measured and, within the Standard Model (SM) framework, precisely predicted. Hence, comparison of experiment and theory tests the SM at its quantum loop level. A deviation in $a_\mu^{\rm exp}$ from the SM expectation would signal effects of new physics, with current sensitivity reaching up to mass scales of $\mathcal{O}$(TeV) [1]. For recent thorough muon $g-2$ reviews, see e.g. Refs. [2–4].

The E821 experiment at Brookhaven National Lab (BNL) studied during the years 1997−2001 the precession of μ^+ and μ^- in a constant external magnetic field as they circulated in a confining storage ring. It found[2] [6]

$$\begin{aligned} a_{\mu^+}^{\rm exp,BNL} &= 116\,592\,040(60)(50) \times 10^{-11} \,, \\ a_{\mu^-}^{\rm exp,BNL} &= 116\,592\,150(80)(30) \times 10^{-11} \,, \end{aligned} \tag{56.2}$$

where the first errors are statistical and the second systematic. Assuming CPT invariance and taking into account correlations between systematic uncertainties, one finds for their average [5, 6]

$$a_\mu^{\rm exp,BNL} = 116\,592\,089(54)(33) \times 10^{-11} \ . \tag{56.3}$$

These results represent about a factor of 14 improvement over the classic CERN experiments of the 1970s [7].

Further improvement of the measurement by a factor of four by setting up the E821 storage ring at the Fermilab National Accelerator Laboratory (FNAL), and utilizing a cleaner and more intense muon beam and improved detectors [8] is in progress. After six runs, the experiment has completed data taking in 2023. A first analysis with positive muons based on a fraction of the data collected in 2018 was released in 2021 confirming the BNL result with similar precision [9]. A second analysis released in 2023 was based on a four times larger positive muon data sample collected in 2019 and 2020, and benefitted from instrumental improvements and reduced systematic uncertainties [10]. The result is dominated by statistical uncertainty and consistent with but more precise than the previous one. Their combination yields [10]

$$a_{\mu^+}^{\rm exp,FNAL} = 116\,592\,055(24) \times 10^{-11} \,. \tag{56.4}$$

The ongoing analysis of the remaining 2021–2023 dataset is expected to further reduce the uncertainty by a factor of two.

The FNAL result agrees with the BNL measurement and has improved precision. Their average assuming CPT invariance reads [10]

$$a_\mu^{\rm exp} = 116\,592\,059(22) \times 10^{-11} \ , \tag{56.5}$$

[1] Also referred to as *anomalous magnetic moment* despite being dimensionless.

[2] The results reported by the experiment have been updated in Eqs. (56.2) and (56.3) to the newest value for the absolute muon-to-proton magnetic ratio $\lambda = 3.183\,345\,142(71)$ [5]. The change induced in $a_\mu^{\rm exp}$ with respect to the value of $\lambda = 3.183\,345\,39(10)$ used in Ref. [6] amounts to $+10 \times 10^{-11}$.

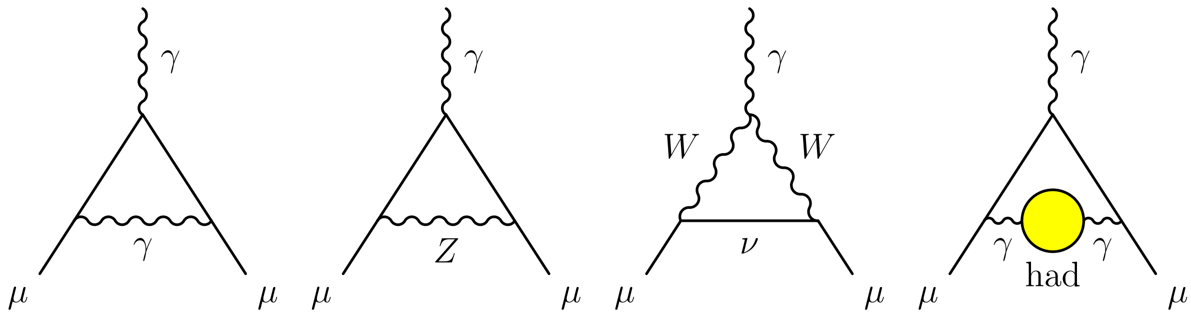

Figure 56.1: Representative diagrams contributing to $a_\mu^{\rm SM}$. From left to right: first order QED (Schwinger term), lowest-order weak, lowest-order hadronic.

providing a relative precision of 0.19 parts per million.

Another muon $g-2$ experiment with similar sensitivity but using an alternative zero-electric-field technique with a low-emittance and low-momentum muon beam is currently under construction at J-PARC in Japan [11].

The SM prediction $a_\mu^{\rm SM}$ is generally divided into three parts (see Fig. 56.1 for representative Feynman diagrams)

$$a_\mu^{\rm SM} = a_\mu^{\rm QED} + a_\mu^{\rm EW} + a_\mu^{\rm Had}\ . \tag{56.6}$$

In the following discussion we use the numerical estimates provided by the Muon $g-2$ Theory Initiative White Paper [4].

The QED part includes all photonic and leptonic (e,μ,τ) loops starting with the classic $\alpha/2\pi$ Schwinger contribution [12]. It has been computed through five loops [4, 13, 14]

$$\begin{aligned} a_\mu^{\rm QED} =& \frac{\alpha}{2\pi} + 0.765\,857\,420(13)\left(\frac{\alpha}{\pi}\right)^2 + 24.050\,509\,85(23)\left(\frac{\alpha}{\pi}\right)^3 \\ &+ 130.8782(60)\left(\frac{\alpha}{\pi}\right)^4 + 751.0(9)\left(\frac{\alpha}{\pi}\right)^5 + \cdots \end{aligned} \tag{56.7}$$

with little change in the coefficients since our last update of this review. Employing $\alpha^{-1} = 137.035\,999\,046(27)$, obtained from the precise measurements of $h/m_{\rm Cs}$ [15], the Rydberg constant, and $m_{\rm Cs}/m_e$ leads to[3] [13]

$$a_\mu^{\rm QED} = 116\,584\,718.93(0.10)\times 10^{-11}\,, \tag{56.8}$$

where the small error results mainly from the uncertainties in the estimate of the six loop contribution and in α.

Loop contributions involving heavy $W^\pm, Z$ or Higgs particles are collectively labeled as $a_\mu^{\rm EW}$. They are suppressed by at least a factor of $(\alpha/\pi)\cdot(m_\mu^2/m_W^2) \simeq 4\times 10^{-9}$. At 1-loop order [19]

$$\begin{aligned} a_\mu^{\rm EW}[\text{1-loop}] &= \frac{G_\mu m_\mu^2}{8\sqrt{2}\pi^2}\left[\frac{5}{3} + \frac{1}{3}\left(1-4\sin^2\theta_{\rm W}\right)^2 + \mathcal{O}\left(\frac{m_\mu^2}{M_W^2}\right) + \mathcal{O}\left(\frac{m_\mu^2}{m_H^2}\right)\right] \\ &= 194.8\times 10^{-11}\,, \end{aligned} \tag{56.9}$$

[3] A recent measurement using Rb-87 atoms [16] resulted in $\alpha^{-1} = 137.035\,999\,206(11)$, which exhibits a larger than 5σ discrepancy with the Cs-133 based result. Using this value in Eq. (56.7) leads to a reduction of $a_\mu^{\rm QED}$ by 0.14×10^{-11}, which is larger than the quoted uncertainty in Eq. (56.8), but still negligible compared to other uncertainties affecting $a_\mu^{\rm SM}$. This discrepancy impacts, however, the SM prediction of the magnetic moment of the electron [14, 17], which differs by respectively -3.9σ and $+2.1\sigma$ from the most recent experimental value [18], depending on whether the Cs-133 or Rb-87 based α value is used.

for $\sin^2\theta_W \equiv 1 - M_W^2/M_Z^2 \simeq 0.223$, and where $G_\mu \simeq 1.166 \times 10^{-5}$ GeV^{-2} is the Fermi coupling constant. Two-loop corrections are relatively large and negative [20]. For a Higgs boson mass of 125 GeV it amounts to $a_\mu^{\rm EW}[\text{2-loop}] = -41.2(1.0) \times 10^{-11}$ [20], where the uncertainty stems from quark triangle loops. The 3-loop leading logarithms are negligible, $\mathcal{O}(10^{-12})$ [20, 21]. Overall one finds

$$a_\mu^{\rm EW} = 153.6(1.0) \times 10^{-11} \,. \tag{56.10}$$

A recent complete 2-loop numerical evaluation of the electroweak correction [22] when adjusted for appropriate light quark masses confirmed the result in Eq. (56.10).

Hadronic (quark and gluon) loop contributions to $a_\mu^{\rm SM}$ give rise to its main uncertainty. One traditionally relies on a data-driven dispersion relation approach to evaluate the lowest-order $\mathcal{O}(\alpha^2)$ hadronic vacuum polarization contribution $a_\mu^{\rm Had}[\text{LO}]$ from corresponding cross section measurements [23]

$$a_\mu^{\rm Had}[\text{LO}] = \frac{1}{3}\left(\frac{\alpha}{\pi}\right)^2 \int_{m_\pi^2}^{\infty} ds\, \frac{K(s)}{s} R^{(0)}(s)\,, \tag{56.11}$$

where $K(s)$ is a QED kernel function [24], and where $R^{(0)}(s)$ denotes the ratio of the bare[4] cross section for e^+e^- annihilation into hadrons to the pointlike muon-pair cross section at center-of-mass energy $\sqrt{s}$. The function $K(s) \sim 1/s$ in Eq. (56.11) emphasizes the low-energy part of the integral so that $a_\mu^{\rm Had}[\text{LO}]$ is dominated by the $\rho(770) \to \pi^+\pi^-$ resonance.

The analysis of Eq. (56.11) results in the representative value [4, 25–30]

$$a_\mu^{\rm Had}[\text{LO}] = 6\,931(40) \times 10^{-11}\,, \tag{56.12}$$

whose error is dominated by systematic uncertainties in the $e^+e^- \to$ hadrons cross-section data, and also includes a small uncertainty due to perturbative QCD, which is used at intermediate and large energies in the dispersion integral to predict the contribution from the quark-antiquark continuum. The experimental precision of Eq. (56.12) is limited by a discrepancy between the most precise $\pi^+\pi^-$ data from the BABAR and KLOE experiments [27, 29]. Recent preliminary $\pi^+\pi^-$ data from the CMD-3 experiment [31] give a significantly larger contribution to $a_\mu^{\rm Had}[\text{LO}]$ but disagree with essentially all other datasets. New or updated measurements are expected from several experiments during the next years.

Alternatively, one can use precise vector spectral functions from $\tau \to \nu_\tau +$ hadrons decays [32] that can be related to isovector $e^+e^- \to$ hadrons cross sections by isospin symmetry. Analyses replaced e^+e^- data in the two-pion and four-pion channels by the corresponding isospin- transformed τ data, and applied isospin-violating corrections [33]. Owing to the progress in the precision of the e^+e^- data, the τ data are now less precise and less reliable due to additional theoretical uncertainties, so that recent $a_\mu^{\rm Had}[\text{LO}]$ evaluations ignored them.

Progress has been achieved on the evaluation of $a_\mu^{\rm Had}[\text{LO}]$ using lattice QCD, which allows to directly compute the real part of the two-point correlation function without invoking the resonances occurring on the imaginary axis [34–36]. The required sub-percent precision represents a challenge requiring to master systematic and extrapolation uncertainties. A combination of recent lattice QCD evaluations [37, 38] gives [4]

$$a_\mu^{\rm Had}[\text{LO}] = 7\,116(184) \times 10^{-11}\,, \tag{56.13}$$

[4]The bare cross section is defined as the measured cross section corrected for initial-state radiation, electron-vertex loop contributions and vacuum-polarization effects in the photon propagator. However, QED effects in the hadron vertex and final state, as photon radiation, are included.

with an uncertainty that is not yet competitive with the data-driven approach. Another recent lattice QCD calculation with much reduced uncertainty found [39]

$$a_\mu^{\rm Had}[{\rm LO}] = 7\,075(55) \times 10^{-11}\,, \tag{56.14}$$

whose error is dominated by systematic uncertainties. It exhibits a 2.1σ tension with the data-driven result, growing up to 3.7σ for the modified observable $a_{\mu,{\rm win}}$ that has favourable properties for computation on the lattice[5] [38, 39, 41]. Because of the complexity and challenges of such a calculation, it is critical that it is confronted with independent lattice results of comparable precision [38, 41, 42].

Higher order hadronic loop contributions are also obtained from dispersion relations using the same $e^+e^- \to$ hadrons data but with different integration kernels [43]. Recent evaluations found $a_\mu^{\rm Had,Disp}[{\rm NLO}] = (-98.3 \pm 0.7) \times 10^{-11}$ [4, 30] and $a_\mu^{\rm Had,Disp}[{\rm NNLO}] = (12.4 \pm 0.1) \times 10^{-11}$ [44]. In the case of hadronic light-by-light scattering contributions, which enter at NLO, recent studies [3, 4, 45, 46] based on short-distance QCD, dispersion relations and lattice QCD calculations lead to the improved prediction $a_\mu^{\rm Had,LBL}[{\rm NLO}] = 92(18) \times 10^{-11}$ [4].[6] One thus finds for the sum of the three higher order hadronic terms

$$a_\mu^{\rm Had}[{\rm N(N)LO}] = 6(18) \times 10^{-11}\,, \tag{56.15}$$

where the error is dominated by the hadronic light-by-light scattering uncertainty.

Adding Eqs. (56.8), (56.10), (56.12) and (56.15) gives the representative SM prediction

$$a_\mu^{\rm SM} = 116\,591\,810(1)(40)(18) \times 10^{-11}\,, \tag{56.16}$$

whose errors are due to the electroweak, lowest-order hadronic, and higher-order hadronic contributions, respectively. The difference between experiment and theory

$$\Delta a_\mu = a_\mu^{\rm exp} - a_\mu^{\rm SM} = 249(22)(43) \times 10^{-11}\,, \tag{56.17}$$

where the errors are from experiment and theory prediction (with all errors combined in quadrature, taking into account a small anticorrelation [4] between the data-driven lowest and higher order contributions), exhibits a discrepancy of 5.2σ.[7] The new FNAL measurement has increased the discrepancy with the SM prediction from previously 3.7σ (BNL only) and 4.2σ (including the first FNAL result), respectively. Despite the large significance, the situation remains inconclusive due to the discrepancies in the hadronic cross section data underlying the dispersive $a_\mu^{\rm Had}[{\rm LO}]$ determination.

Figure 56.2 represents the difference between experimental and predicted a_μ values based on the most recent data-driven and lattice QCD based evaluations of $a_\mu^{\rm Had}[{\rm LO}]$.

A leading candidate explanation for the deviation observed in Δa_μ has been supersymmetry particle loop contributions. Such a scenario is quite natural, since generically, supersymmetric models predict [1] an additional contribution to $a_\mu^{\rm SM}$

$$a_\mu^{\rm SUSY} \simeq \pm\, 130 \times 10^{-11} \cdot \left(\frac{100~{\rm GeV}}{m_{\rm SUSY}}\right)^2 \tan\beta\,, \tag{56.18}$$

[5] A dispersive approach similar to that for $a_\mu^{\rm Had}[{\rm LO}]$ is used to determine the hadronic contribution to the running of the electromagnetic fine structure constant at the Z-boson mass $\Delta\alpha_{\rm had}(m_Z)$, which is a critical ingredient of the global electroweak fit. One may therfore ask if the goodness of that fit would break down were $a_\mu^{\rm Had}[{\rm LO}]$ in Eq. (56.12) be brought in compatibility with the result (56.14). However, the difference in the relevant integration kernels (whereas 73% of the contribution to $a_\mu^{\rm Had}[{\rm LO}]$ stems from the $\pi^+\pi^-$ channel, it is only 13% in the case of $\Delta\alpha_{\rm had}(m_Z)$) allows to absorb large effects in $a_\mu^{\rm Had}[{\rm LO}]$ without creating a significant tension in the electroweak fit. Detailed discussions are available in the literature [40].

[6] A very recent lattice QCD based calculation of $a_\mu^{\rm Had,LBL}[{\rm NLO}]$ [47], not included in [4], gives a consistent result with [4, 46] using a different approach.

[7] The probabilistic interpretation of the discrepancy requires caution due to the systematic nature of the $a_\mu^{\rm SM}$ uncertainty.

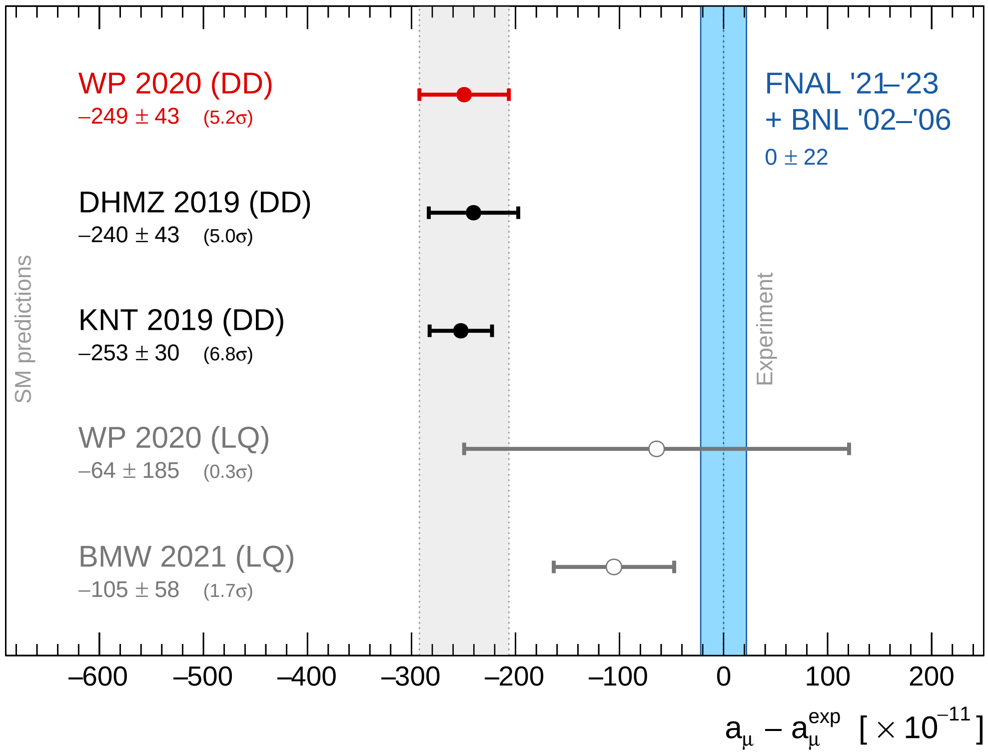

Figure 56.2: Compilation of recent results for a_μ (in units of 10^{-11}), subtracted by the central value of the experimental average (56.3). The light blue vertical band indicates the total experimental uncertainty. The SM predictions shown differ only in the value used for the $a_\mu^{\rm Had}$[LO] contribution. They are taken from: WP 2020 [4] (Eq. (56.16), also indicated by the light grey vertical band), DHMZ 2019 [29], KNT 2019 [30], all three based on the data-driven (DD) $a_\mu^{\rm Had}$[LO] evaluation, and the lattice QCD (LQ) based evaluations WP 2020 [4, 37, 38] and BMW 2021 [39]. Note that the quoted errors in the figure do not include the uncertainty on the subtracted experimental value. To obtain for each theory calculation a result equivalent to Eq. (56.17), the errors from theory and experiment must be added in quadrature.

where $m_{\rm SUSY}$ is a representative supersymmetric mass scale, $\tan\beta \simeq 3$–40 a potential enhancement factor, and ±1 corresponds to the sign of the μ term in the supersymmetric Lagrangian. Supersymmetric particles in the mass range 100–500 GeV could be the source of the deviation Δa_μ. If so, those particles should be directly observable at the Large Hadron Collider at CERN. So far, there is however no direct evidence in support of the supersymmetry interpretation.

New physics effects [1] other than supersymmetry could also explain a non-vanishing Δa_μ. A popular scenario involves the "dark photon", a relatively light hypothetical vector boson from the dark matter sector that couples to our world of particle physics through mixing with the ordinary photon [48–50]. As a result, it couples to ordinary charged particles with strength $\varepsilon \cdot e$ and gives rise to an additional muon magnetic anomaly contribution

$$a_\mu^{\rm dark\ photon} = \frac{\alpha}{2\pi}\varepsilon^2 F(m_V/m_\mu)\,, \tag{56.19}$$

where $F(x) = \int_0^1 2z(1-z)^2/[(1-z)^2 + x^2 z]\,dz$. For values of $\varepsilon \sim 1$–2×10^{-3} and $m_V \sim 10$–$100\,\mathrm{MeV}$, the dark photon, which was originally motivated by cosmology, can provide a viable solution to the muon $g-2$ discrepancy. However, experimental constraints appear to rule out the simplest model in which the dark photon has equal couplings to electrons and muons and decays primarily to e^+e^- pairs [51] or invisible dark particles [52] that give rise to missing event energy. One can expand the dark photon scenario into a more complete theory in which several new particles contribute to the muon $g-2$. Direct searches for a dark photon, therefore, continue to be well motivated, but with primary guidance coming from astrophysics [53].

Recent popular solutions to the muon anomaly discrepancy have also focused on loop contributions induced by relatively light scalar or pseudo scalar particle appendages from physics beyond the SM [54, 55].

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