69. Multibody Charm Analyses

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69.1 Overview

The study of multibody charm decays is a vibrant field, with vast and fast-increasing datasets, new developments in theory and experimental technique, and implications reaching beyond charm. This review is structured as follows: Sec. 69.2 summarizes key aspects of amplitude models that describe multibody charm decays, their application to data, and theory progress; Sec. 69.3 reviews the special role of charm threshold data in model-independent approaches; and Sec. 69.4 describes applications of multi-body charm analyses, focusing on Charge-Parity (CP) violation, charm mixing and the role of threshold data in this context. In Sec. 69.5, we conclude.

69.2 Kinematics & Models

The differential decay rate to a point $\mathbf{s} = (s_1, \ldots, s_n)$ in n dimensional phase space can be expressed as

$$d\Gamma = |\mathcal{M}(\mathbf{s})|^2 \left| \frac{\partial^n \phi}{\partial(s_1 \ldots s_n)} \right| d^n s\,, \tag{69.1}$$

where $|\partial^n \phi / \partial(s_1 \ldots s_n)|$ represents the density of states at $\mathbf{s}$, and $\mathcal{M}$ the matrix element for the decay at that point in phase space, which is $2, 5, 8, \ldots$ dimensional for D decays to $3, 4, 5, \ldots$ spinless particles. Additional parameters are required to fully describe decays involving particles with non-zero spin in the initial or final state.

For the important case of D decays to three pseudoscalars, the decay kinematics can be represented in a two-dimensional Dalitz plot [1]. This is usually parameterized in terms of $s_{12} \equiv (p_1 + p_2)^2$ and $s_{23} \equiv (p_2 + p_3)^2$, where p_1, p_2, and p_3 are the four-momenta of the final-state particles. In terms of these variables, phase-space density is constant across the kinematically allowed region, so that any structure seen in the Dalitz plot is a direct consequence of the dynamics encoded in $|\mathcal{M}|^2$. Note that decays to four or more particles cannot be unambiguously described in terms of analogously-defined variables s_{ij} and s_{ijk}, as these cannot describe parity-odd kinematics; these are absent in the three-body case because the decay products are confined to a plane. See Sec. 69.4.4 for a discussion on the use of parity-odd variables in four-body decays.

In the widely-used isobar approach, the matrix element $\mathcal{M}$ is modeled as a sum of interfering decay amplitudes, each proceeding through resonant two-body decays [2–6]. See [2, 7, 8] for a review of resonance phenomenology. In most analyses, each resonance is described by a relativistic Breit–Wigner [9] or Flatté [10] lineshape, and the model includes a non-resonant term with a constant phase and magnitude. This approach has well-known theoretical limitations, such as the violation of unitarity and analyticity, which can break the relationship between magnitude and phase across phase space. This motivates the use of more sophisticated descriptions, especially for broad, overlapping resonances (frequently found in S-wave components) where these limitations are particularly problematic. In charm analyses, these approaches have included the K-matrix [11] which respects two-body unitarity; the use of LASS scattering data [12]; dispersive methods [13–16]; multi-meson models using chiral Lagrangians [17–20]; and methods based on chiral unitarity [21–24]; QCD factorization [25–28]; and quasi model-independent parametrizations that use generic lineshapes, with minimal theory input and many free parameters, for a subset of resonances [29–37].

An important example of a multibody charm decay is $D^0 \to K^0_S \pi^+ \pi^-$, which is a key channel in CP violation and charm-mixing analyses. A joint analysis by BaBar and Belle [38] describes a Dalitz plot with 1.1M $D^0 \to K^0_S \pi^+ \pi^-$ events (the largest sample analyzed in this way) with 18 resonant components, including four doubly Cabibbo suppressed ones. The model uses a K-matrix description for the $\pi\pi$ S-wave based on [39] and input from LASS scattering data [12, 40] for the $K\pi$ S-wave, with no need to add a non-resonant component to describe the data. These data were reanalyzed by [25] in a QCD factorization framework, using line-shape parametrizations for the S [41, 42] and P wave [14] contributions that preserve two-body unitarity and analyticity. The analyses give compatible results for the components they share. Recently, BESIII studied both the $D^0 \to K^0_S\pi^+\pi^-$ and the $D^0 \to K^0_L\pi^+\pi^-$ Dalitz plot, providing insights into U-spin breaking effects [43].

The field of charm amplitude analyses remains very active. Publications since the last update of this review two years ago include Dalitz plot analyses of $D^+ \to \pi^+\eta\eta$, $D^+_s \to K^0_S K^0_L \pi^+$, $D^+ \to K^0_S K^0_S \pi^+$, $D^+ \to K^0_S \pi^+\eta$, $D^0 \to \pi^+\pi^-\eta$, $D^+ \to \pi^+\pi^0\eta$, by BESIII [44–48] and $D^+_s \to \pi^-\pi^+\pi^+$ by LHCb [36]. LHCb performed an amplitude analyses of the decay of an excited charm state, $D_{s1}(2460)^+ \to D^+_s\pi^+\pi^-$ [49]. The amplitude structure of charm baryon decays was studied by BESIII in $\Lambda_c \to \Lambda\pi^+\eta$ and by LHCb in $\Lambda_c \to pK^-\pi^+$ [50, 51] and $\Xi^+_c \to pK^-\pi^+$ [52]. BESIII continued its series of four-body amplitude analyses, investigating the hadronic decays $D^+ \to K^-\pi^+\pi^+\pi^0$, $D^+_s \to \pi^+\pi^+\pi^-\pi^0$, $D^0 \to \pi^+\pi^-\pi^+\pi^-$, and $D^0 \to \pi^+\pi^-\pi^0\pi^0$ [53–55] as well as the semileptonic decays $D^+ \to K^0_{\mathrm{S}}\pi^0\mu^+\nu_\mu$, $D^+ \to K^0_{\mathrm{S}}\pi^0 e^+\nu_e$, $D^0 \to \overline{K}^0\pi^-\mu^+\nu_\mu$, $D^0 \to \overline{K}^0\pi^- e^+\nu_e$, $D^+ \to \pi^+\pi^- e^+\nu_e$, and $D^+_s \to K^+K^-\mu^+\nu_\mu$ [56–61]. BESIII published five-body amplitude analyses of the decays $D^+_s \to \pi^+\pi^+\pi^-\pi^0\pi^0$, $D^+ \to K^-\pi^+\pi^0 e^+\nu_e$, and $D^0 \to K^-\pi^+\pi^- e^+\nu_e$. [62, 63].

Noteworthy is the increasing sophistication of recent amplitude analyses, most of which go substantially beyond the isobar model with Breit–Wigner and Flatté lineshapes. However, with the notable exceptions of [15, 16, 23, 24, 64, 65], they remain within the isobar framework, which describes the decay as a series of two-body processes. This approach ignores long-range hadronic effects such as re-scattering, and does not respect three (or four)-body unitarity and analyticity. Recent quasi model-independent measurements of the $\pi^+\pi^-$ lineshape emphasize the importance of such effects. Significant differences have been found between the $\pi^+\pi^-$ S-wave components observed in $D^+_s \to \pi^+\pi^-\pi^+$ [31, 35, 36] and $D^+ \to \pi^+\pi^-\pi^+$ [37], and between either of them and phase shifts observed in scattering data [66–68].

Several groups work on improved models and their application to data. Dispersive techniques, which respect three-body unitarity and analyticity, have been applied to regions of the $D^+ \to K^-\pi^+\pi^+$ and $D^+ \to K^0_S\pi^0\pi^+$ Dalitz plots below the $\eta' K$ threshold [15, 16], where they provide a good description of the data with fewer fit parameters than the isobar approach. Reference [64] uses a unitary coupled channel approach to describe $D^+ \to K^-\pi^+\pi^+$, which has no restrictions on the kinematic range, but requires additional parameters to describe the Dalitz plot above the $\eta' K$ threshold. Using an effective chiral Lagrangian, the authors of reference [19] provide a description of the annihilation contribution to the decay amplitude which respects three-body unitarity. The approach provides a good description of LHCb's $D^+ \to K^+K^-K^+$ data, with fewer parameters than an equivalent isobar model [65]. The same channel has more recently been re-analyzed by the authors of [23], who argue that the internal emission diagram should dominate and use a chiral unitarity-based approach to achieve a reasonable description of the data with two free parameters.

Limitations in the theoretical description of interfering resonances are the leading source of systematic uncertainty in many analyses. In some cases, the model uncertainty can be removed through model-independent methods, often relying on input from the charm threshold, as discussed in Sec. 69.3. The authors of [69] expand the scope and applicability of the quasi model-independent approach in amplitude fits. At the same time, increasingly sophisticated models are being developed, and applied to data. The authors of [20] and [70] provide valuable practical frameworks for sophisticated amplitude analyses.

69.3 Model Independent Methods and the Charm Threshold

Precision measurements of the CP-violation phase γ/ϕ_3 using $B^- \to DK^-$, $D \to K^0_S\pi^+\pi^-$ and related decay modes, as well as precision measurements of charm-mixing parameters in decay modes such as $D^0 \to K^0_S\pi^+\pi^-$, require input on the relative decay amplitudes of D^0 and $\overline{D}^0$. In the above examples, $K^0_S\pi^+\pi^-$ can be replaced with other final states accessible to both D^0 and $\overline{D}^0$ [71–79]. While the magnitudes of the decay amplitudes can be measured precisely using the vast charm samples at the B factories and LHCb, obtaining the relative phases requires either amplitude models with reliable phase motion, or model-independent approaches. More details on the measurements of γ/ϕ_3 and charm mixing can be found in sections 69.4.1 and 69.4.2.

Model-independent measurements of relative phases between D^0 and $\overline{D}^0$ decay amplitudes rely on interference effects in the decays of well-defined coherent superpositions of D^0 and $\overline{D}^0$. These are accessible at the charm threshold [74, 77, 79–84], where CLEO–c and BESIII operate. There, quantum-correlated D-meson pairs are produced, which have opposite CP and flavor content. For example, if one (the tag) is identified as a CP eigenstate through its decay (e.g. to K^+K^- or $K_S\pi^0$), the other D meson is an eigenstate state with the opposite CP eigenvalue. (For a discussion of other potential sources of correlated charm with different characteristics, see [85].) The decay rate of a CP eigenstate $\frac{1}{\sqrt{2}}\left(D^0 \pm \overline{D}^0\right)$ is proportional to $|A_D|^2 + |\overline{A}_D|^2 \pm 2|A_D||\overline{A}_D|\cos(\delta_D)$, where A_D and $\overline{A}_D$ are the D^0 and $\overline{D}^0$ decay amplitudes, and δ_D is their sought-after phase difference. Other quantum correlations give access to $\sin(\delta_D)$. For multibody decays, A_D, $\overline{A}_D$ and δ_D vary across phase space. Model-independent phase information is measured either integrated over the entire phase space of the decay, or in subregions/bins. The results can be expressed in terms of one complex parameter $\mathcal{Z} = Re^{-i\delta_D} = c + is$ per pair of CP-conjugate bins; c and s are the average of $\cos(\delta_D)$ and $\sin(\delta_D)$ over that region of phase space. Amplitude models can be used to optimize the binning for sensitivity to γ/ϕ_3, without introducing a model-dependent bias in the result [86]; novel unbinned approaches have the potential to further increase the precision on γ/ϕ_3 in future analyses [87–89].

BESIII data have been analyzed to provide binned $\mathcal{Z}$ for $K_{S,L}\pi^+\pi^-$, $K_{S,L}K^+K^-$, $K^+\pi^-\pi^0$, $\pi^+\pi^-\pi^+\pi^-$, $K^+K^-\pi^+\pi^-$, and $K^+\pi^-\pi^+\pi^-$ final states [90–96]. Many of these analyses were first carried out with CLEO–c data [97–102].

For self-conjugate decays such as $D^0 \to \pi^+\pi^-\pi^0$, one can define the CP-even fraction F_+ which is +1 for a CP-even eigenstate and 0 for a CP-odd one [82]. F_+ is related to $Re(\mathcal{Z}) = c$, defined for a single CP-conjugate bin pair. For the purpose of γ/ϕ_3 measurements, this information allows the decay to be treated equivalently to a two-body decay, where γ/ϕ_3 is constrained from decay rates without the need to resolve the phase-space distribution. In the mathematical expressions for the decay rates, a factor $(2F_+ - 1)$ multiplies the γ/ϕ_3-sensitive term, leading to the best sensitivity for $F_+ = 1$ or $F_+ = 0$. The decay $D^0 \to \pi^+\pi^-\pi^0$ is found to be, with $F_+ = (94.2 \pm 0.7)\%$, very close to being completely CP-even [83, 103]. The decays $D^0 \to \pi^+\pi^-\pi^+\pi^-$ [83, 94, 99, 104] and $D^0 \to K^+K^-\pi^+\pi^-$ [95, 105] are predominantly CP even with F_+ of 75%, as is $D^0 \to K^+K^-\pi^0$ with 73% [82, 83]; on the other hand, $D^0 \to K^0_S\pi^+\pi^-\pi^0$ is predominantly CP odd with $F_+ = 23\%$ [100,106]; these measurements have percent-level uncertainties. Additional measurements for decays to four, five, and six pions can be found in [107].

Comparing measured parameters like F_+ and $\mathcal{Z}$ with those predicted from amplitude models provides a test of the models' phase motion. The measured value for F_+ in $D^0 \to \pi^+\pi^-\pi^+\pi^-$ for example, $F_+ = (74.6 \pm 0.8)\%$ [99, 104], compares well with $(72.9 \pm 2.0)\%$ calculated from [32]. The coherence factor $|\mathcal{Z}^{3\pi}| = R^{K3\pi} = 0.430^{+0.043}_{-0.039}$ measured in $D^0 \to K^\pm\pi^\mp\pi^\pm\pi^\mp$ quantum correlated events at BESIII and CLEO–c [96, 102] and charm mixing at LHCb [108, 109], compares well with 0.459 ± 0.025 predicted by [33]. The bin-wise comparisons between model and data for

$D^0 \to K^+K^-\pi^-\pi^+$, $\pi^+\pi^-\pi^+\pi^-$, $K_{L,S}\pi^+\pi^-$, $K^\pm\pi^\mp\pi^\pm\pi^\mp$, $K^\pm\pi^\mp\pi^0$, and $K_S^0K^+K^-$ can be found in [43, 91–96, 96, 99] and mostly show reasonable or good agreement. These results are a welcome surprise given the preceding discussion on the theoretical shortcomings of amplitude models.

69.4 Applications of multibody charm analyses for *CP* violation and mixing

Amplitude analyses provide sensitivity to both relative magnitudes and phases of the interfering decay amplitudes. It is especially the sensitivity to phases that makes amplitude analyses such a uniquely powerful tool for studying a wide range of phenomena. Here we concentrate on their critical role in *CP* violation and mixing measurements. Properties of light-meson resonances determined in D amplitude analyses are reported in the light-unflavored-meson section of this *Review*.

The closely related topics of multibody charm decays in measurements of *CP* violation in beauty decays to charm and charm mixing will be discussed in turn, followed by a review of searches for time-integrated *CP* violation in multibody charm decays.

69.4.1 *CP violation in decays of Beauty to Charm*

Neutral D mesons originating from $B^- \to DK^-$ (here denoted as D_{B^-}) are a superposition of D^0 and $\overline{D}^0$ with a relative phase that depends on the CKM unitarity triangle parameter γ/ϕ_3,

$$D_{B^-} \propto D^0 + r_B e^{i(\delta_B - \gamma)}\overline{D}^0,$$

where δ_B is a *CP* conserving strong phase, and $r_B \sim 0.1$ is the magnitude of the ratio of the $B^- \to \overline{D}^0K^-$ and $B^- \to D^0K^-$ decay amplitudes. In the corresponding *CP*-conjugate expression, γ/ϕ_3 changes sign. The amplitude analysis of the subsequent $D_{B^\pm}$ decay to a state accessible to both D^0 and $\overline{D}^0$ allows the measurement of γ/ϕ_3 [71–78]. The method generalizes to similar B hadron decays, such as $B^0 \to DK^{*0}$, and, albeit with reduced sensitivity, to decays where the kaon is replaced by a pion.

Measurements based on this technique have been reported by BaBar [110, 111], Belle [112–114], Belle II [115, 116], and LHCb [117–136]. Since it relies on $D^0 - \overline{D}^0$ interference, the phase differences between the relevant D^0 and $\overline{D}^0$ decay amplitudes are required in order to extract γ/ϕ_3. Because the theoretical shortcomings of amplitude models, discussed above, make their phases unreliable, most recent γ/ϕ_3 measurements use amplitude model-independent approaches to obtain this information [113, 114, 116–121, 123–129, 132, 135]. These depend critically on input from the charm threshold described in Sec. 69.3. Charm mixing also results in a (time-dependent) $D^0 - \overline{D}^0$ superposition, which can be used to measure the relevant phase information as input to γ/ϕ_3 measurements [137, 138]. This method is particularly powerful in doubly Cabibbo-suppressed decays such as $D^0 \to K^+\pi^-\pi^+\pi^-$, and when used in combination with threshold data.

Under some circumstances, with large data sets, the relevant strong phases and γ/ϕ_3 can be extracted without external input, for example in simultaneous analysis of the $B^0 \to DK^+\pi^-$ Dalitz plot and that of the subsequent $D \to K_S^0\pi^+\pi^-$ decay [139]. However, the global effort to measure γ/ϕ_3 to sub-degree precision will continue to rely critically on input from the charm threshold and the analysis of BESIII's full dataset [140].

The most precise individual γ/ϕ_3 measurement is $\gamma/\phi_3 = 68.5^{\circ}{}^{+5.2^\circ}_{-5.1^\circ}$ resulting from LHCb's amplitude model-independent study of $D_{B^-} \to K_S^0\pi^+\pi^-$ and $D_{B^-} \to K_S^0K^+K^-$ [141], using input from the charm threshold [90, 92, 97]. LHCb's analysis of $D_{B^-} \to K^+\pi^-\pi^+\pi^-$ decays leads to the second most precise individual γ/ϕ_3 measurement with $\gamma/\phi_3 = 54.8^{\circ}{}^{+6.0^\circ}_{-5.8^\circ} \pm 0.6^{\circ}{}^{+6.7^\circ}_{-4.3^\circ}$ [128]. This analysis benefits from an optimized model-informed partition of the five-dimensional $D^0 \to K^+\pi^-\pi^+\pi^-$ phase space [33, 102], and input from the charm threshold [96, 102]; the third uncertainty on γ/ϕ_3 in the result quoted above quantifies the impact of the uncertainty on the charm parameters measured at the threshold. These have recently been constrained further from charm mixing measurements [108, 109], leading to a significant improvement in the precision of γ/ϕ_3 [135].

The current world average on γ/ϕ_3 [142, 143] is dominated by LHCb who finds, combining its results with input from CLEO–c and BESIII in an amplitude-model-independent approach, $\gamma = 62.8^\circ \pm 2.6^\circ$ [135]. Belle II is also expected to reach excellent precision in γ/ϕ_3, benefiting from its strength in reconstructing decays with neutral particles in the final state. Combining Belle and Belle II data with charm threshold input, the collaborations find an average, dominated by model-independent results, of $\phi_3 = 75.2^\circ \pm 7.6^\circ$ [115].

The interference between mixing and decay in $B^0 \to D^0 h^0$ with $h^0 = \pi^0, \eta, \omega$ provides sensitivity to another CKM unitarity triangle parameter, β/ϕ_1, which can be extracted from the Dalitz plot of the subsequent $D^0 \to K^0_S\pi^+\pi^-$ decay [38, 144–147]. The combined BaBar/Belle analysis based on this technique resolved the ambiguity in β/ϕ_1 present in other measurements [38].

Further details on *CP* violation in beauty and charm can be found in [148, 149].

69.4.2 *Charm Mixing and CP violation*

Time-dependent amplitude analyses in decays to final states that are accessible to both D^0 and $\overline{D}^0$ have unique sensitivity to mixing parameters. A Dalitz plot analysis of a self-conjugate final state, such as $K^0_S\pi^+\pi^-$ and $K^0_SK^+K^-$, allows a direct measurement of x and y, the normalized mass and width difference of the $D^0 - \overline{D}^0$ system's mass eigenstates [150]. This is in contrast to decays like $D^0 \to K^+\pi^-$, which only provide access to decay-specific parameters x'^2, y'.

The phase differences between D^0 and $\overline{D}^0$ amplitudes across the Dalitz plot affect these measurements in the same way as those of γ/ϕ_3 in $B^+ \to DK^+$, and can be taken into account in an amplitude model-independent way using the same charm threshold results [74, 79–84, 148]. This approach resulted in the first observation of a non-zero mass difference between the neutral charm mass eigenstates by LHCb [151], using a model-independent analysis of $D^0 \to K^0_S\pi^+\pi^-$. Applying the same technique to their combined dataset, Belle and Belle II find consistent results, albeit with lower precision [152]. The "bin flip" technique applied in these measurements exploits that *CP*-conjugate regions of the $D^0 \to K^0_S\pi^+\pi^-$ Dalitz plot have near-identical experimental efficiencies [153]. This technique is also sensitive to *CP* violation in mixing and in the interference between mixing and decay, which is discussed further in [148, 149].

The simultaneous dependence on charm mixing parameters and the same relative phases that affect the measurement of γ/ϕ_3 can also be exploited to constrain those phases [137, 138], using mixing parameters as external input. This led to an improvement in the measurement of the $D^0 \to K^+\pi^-\pi^+\pi^-$ coherence factor and relative strong phase from $R^{K3\pi} = 0.52^{+0.10}_{-0.09}$, $\delta^{K3\pi} = 167^{\circ}{}^{+31^\circ}_{-19^\circ}$ measured at the charm threshold [96, 102] to $R^{K3\pi} = 0.430^{+0.043}_{-0.039}$, $\delta^{K3\pi} = 163^{\circ}{}^{+13.8^\circ}_{-14.8^\circ}$ when combined with LHCb charm mixing data [108, 109]. It is worth noting, though, that charm mixing alone is only sensitive to a linear combination of $R^{K3\pi}\cos(\delta^{K3\pi})$ and $R^{K3\pi}\sin(\delta^{K3\pi})$ - combining this with constraints from the threshold allows a measurement in terms of $R^{K3\pi}$ and $\delta^{K3\pi}$.

69.4.3 *Everything, Everywhere, All at Once*

The discussion above shows how closely related the measurements of γ/ϕ_3, charm mixing, and the analysis of charm threshold data are, due to the shared charm parameters. While above, the emphasis was on the importance of charm as input to γ/ϕ_3, and threshold data as input to charm mixing and γ/ϕ_3, these relationships are in fact omnidirectional. This is demonstrated by LHCb's combined analysis on beauty and charm data [135], as well as a global combination by UTfit members and collaborators [143]. Not only does charm input improve the precision on γ/ϕ_3, both analyses also show that input from beauty decays to charm (such as $B^- \to DK^-$) has a significant impact on charm parameters.

69.4.4 *Searches for time-integrated CPV in charm*

The recent observation of direct CP violation in LHCb's analysis of $D^0 \to \pi^+\pi^-$ and $D^0 \to K^+K^-$ decays [154, 155], and the open question as to its Standard Model or beyond the Standard Model nature, add renewed interest in the search for CP violation in other decays.

Multibody decays, with their rich structure and varying strong phase across phase space, could potentially have particularly high sensitivity to CP violation. Comparing the results of amplitude fits for CP-conjugate decay modes provides a measure of CP violation, as for example done for $D^0 \to K^0_S K^\pm \pi^\mp$, $D^0 \to K^+K^-\pi^+\pi^-$, and $D^0 \to \pi^+\pi^-\pi^+\pi^-$ using LHCb and CLEO–c data [32, 156–158]. A widely-used amplitude model-independent technique to search for local CP violation in multibody phase space is based on performing a χ^2 comparison of CP-conjugate phase-space distributions. This method was pioneered by BaBar [159] and developed further in [160–162], with results reported by BaBar and LHCb in $D^\pm \to K^+K^-\pi^\pm$ [159, 163–166], CDF in $D^0 \to K^0_S\pi^+\pi^-$ [167], and LHCb in $D^0 \to K^0_S K^\pm\pi^\mp$, $D^+_{(s)} \to K^+K^-K^+$, $D^+ \to \pi^-\pi^+\pi^+$, $D^0 \to K^+K^-\pi^+\pi^-$, $D^0 \to \pi^+\pi^-\pi^+\pi^-$, and $\Xi^+_c \to pK^-\pi^+$ [162, 168–171]. Unbinned methods can increase sensitivity [172, 173] and have been applied by LHCb to $D^+ \to \pi^-\pi^+\pi^+$, $D^0 \to \pi^+\pi^-\pi^0$, $D^0 \to \pi^+\pi^-\pi^+\pi^-$, $D^0 \to K^0_S K^\pm\pi^\mp$, and $\Xi^+_c \to pK^-\pi^+$ [168, 170, 171, 174–176].

An alternative model-independent approach is based on observables in four body decays that are odd under motion reversal ("naïve T") [177–185], which is equivalent to P for scalar particles [185]. One such observable is $C_T = \vec{p}_2 \cdot (\vec{p}_3 \times \vec{p}_4)$, where $\vec{p}_i$ is the i^{th} decay product's three momentum in the decay's center of mass frame. Comparing the P violating asymmetry $A_T \equiv \frac{\Gamma(C_T>0)-\Gamma(C_T<0)}{\Gamma(C_T>0)+\Gamma(C_T<0)}$ with its C-conjugate provides sensitivity to CP violation. Additional variables are proposed in [185], which also provides an excellent review of this topic. Searches for CP violation using P-odd variables have been carried out by BaBar, Belle, Belle II, FOCUS, and LHCb in $D^+ \to K^0_S K^-\pi^+\pi^+$,$D_s \to K^0_S K^-\pi^+\pi^+$ [186]; $D^0 \to K^+K^-\pi^+\pi^-$ [187–190]; $D^+ \to K^+K^0_S\pi^+\pi^-$, $D^+_s \to K^+K^0_S\pi^+\pi^-$ [191]; $D^0 \to K^0_S\pi^+\pi^-\pi^0$ [192]; and $D^0 \to \pi^+\pi^-\pi^+\pi^-$ [175].

LHCb's angular analysis of the rare decays $D^0 \to \pi^+\pi^-\mu^+\mu^-$, $D^0 \to K^+K^-\mu^+\mu^-$, and $\Lambda^+_c \to p\mu^+\mu^-$ yields several observables with sensitivity to physics beyond the Standard Model, including CP-violating ones. The results are in agreement with Standard-Model predictions [193–195].

All results of analyses mentioned in this section are compatible with CP conservation. The observation of CP violation in two body charm decays [154, 155], and the vast data samples about to be collected, provide grounds for optimism that this may change in the foreseeable future.

69.5 Summary

Multibody charm decays offer a rich phenomenology, including unique sensitivity to CP violation and charm mixing. This is a highly dynamic field with many new results (some of which we presented here) and rapidly increasing, high quality datasets. These datasets constitute a huge opportunity, but also a challenge to improve the theoretical descriptions of soft hadronic effects in multibody decays. For some measurements, model-independent methods, many relying on input from the charm threshold, provide a way of removing model-induced uncertainties. At the same time, substantial progress in the theoretical description of multibody decays is being made.

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