

## 60. Quark Masses

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### 60.1 Introduction

This note discusses some of the theoretical issues relevant for the determination of quark masses, which are fundamental parameters of the Standard Model of particle physics. Unlike the leptons, quarks are confined inside hadrons and are not observed as physical particles. Quark masses therefore cannot be measured directly, but must be determined indirectly through their influence on hadronic properties. Although one often speaks loosely of quark masses as one would of the mass of the electron or muon, any quantitative statement about the value of a quark mass must make careful reference to the particular theoretical framework that is used to define it. It is important to keep this *scheme dependence* in mind when using the quark mass values tabulated in the data listings.

Historically, the first determinations of quark masses were performed using quark models. These are usually called constituent quark masses and are of order 350 MeV for the  $u$  and  $d$  quarks. Constituent quark masses model the effects of dynamical chiral symmetry breaking discussed below, and are not directly related to the quark mass parameters  $m_q$  of the QCD Lagrangian of Eq. (60.1). The resulting masses only make sense in the limited context of a particular quark model, and cannot be related to the quark mass parameters,  $m_q$ , of the Standard Model. In order to discuss quark masses at a fundamental level, definitions based on quantum field theory must be used, and the purpose of this note is to discuss these definitions and the corresponding determinations of the values of the masses.

### 60.2 Mass parameters and the QCD Lagrangian

The QCD [1] Lagrangian is

$$\mathcal{L} = \sum_{q=u,d,s,\dots,t} \bar{q} (i\mathcal{D} - m_q) q - \frac{1}{2} \text{tr} G_{\mu\nu} G^{\mu\nu}, \quad (60.1)$$

where the sum runs over the quark flavors  $u$ ,  $d$ ,  $s$ ,  $c$ ,  $b$  and  $t$ .  $\mathcal{D} = (\partial_\mu - igA_\mu) \gamma^\mu$  is the gauge covariant derivative,  $A_\mu$  is the SU(3)-valued gluon field,  $G_{\mu\nu} = \frac{i}{g} [D_\mu, D_\nu]$  is the gluon field strength,  $m_q$  is the mass parameter of quark flavor  $q$ , and  $q$  is the quark Dirac field. After renormalization, the QCD Lagrangian Eq. (60.1) gives finite values for physical quantities, such as scattering amplitudes. Renormalization is a procedure that invokes a subtraction scheme to render the amplitudes finite, and requires the introduction of a dimensionful scale parameter  $\mu$ . The mass parameters in the QCD Lagrangian Eq. (60.1) depend on the renormalization scheme used to define the theory, and also on the scale parameter  $\mu$ . The most commonly used renormalization scheme for QCD perturbation theory is the  $\overline{\text{MS}}$  scheme.

The QCD Lagrangian has a chiral symmetry in the limit that the quark masses vanish. This symmetry is spontaneously broken by dynamical chiral symmetry breaking, and explicitly broken by the quark masses. The nonperturbative scale of dynamical chiral symmetry breaking,  $\Lambda_\chi$ , is around 1 GeV [2]. It is conventional to call quarks heavy if  $m_q > \Lambda_\chi$ , so that explicit chiral symmetry breaking dominates ( $c$ ,  $b$ , and  $t$  quarks are heavy), and light if  $m_q < \Lambda_\chi$ , so that spontaneous chiral symmetry breaking dominates (the  $u$  and  $d$  are light and the  $s$  is considered to be light when using SU(3) $_L$  × SU(3) $_R$  chiral perturbation theory). The determination of light- and heavy-quark masses is considered separately in Sec. 60.5 and Sec. 60.6 below.

At high energies or short distances, nonperturbative effects, such as chiral symmetry breaking, become small and one can, in principle, determine quark masses by analyzing mass-dependent

effects using QCD perturbation theory. Such computations are conventionally performed using the  $\overline{\text{MS}}$  scheme at a scale  $\mu \gg \Lambda_\chi$ , and give the  $\overline{\text{MS}}$  “running” mass  $\overline{m}(\mu)$ . We use the  $\overline{\text{MS}}$  scheme when reporting quark masses; one can readily convert these values into other schemes using perturbation theory.

The  $\mu$  dependence of  $\overline{m}(\mu)$  at short distances can be calculated using the renormalization group (RG) equation,

$$\mu^2 \frac{d\overline{m}(\mu)}{d\mu^2} = -\gamma(\overline{\alpha}_s(\mu)) \overline{m}(\mu), \quad (60.2)$$

where  $\gamma$  is the anomalous dimension which is now known to four-loop order in perturbation theory [3] [4].  $\overline{\alpha}_s$  is the coupling constant [1] in the  $\overline{\text{MS}}$  scheme. Defining the expansion coefficients  $\gamma_r$  by

$$\gamma(\overline{\alpha}_s) \equiv \sum_{r=1}^{\infty} \gamma_r \left( \frac{\overline{\alpha}_s}{4\pi} \right)^r, \quad (60.3)$$

the first four coefficients are given by

$$\begin{aligned} \gamma_1 &= 4, \\ \gamma_2 &= \frac{202}{3} - \frac{20N_L}{9}, \\ \gamma_3 &= 1249 + \left( -\frac{2216}{27} - \frac{160}{3}\zeta(3) \right) N_L - \frac{140}{81}N_L^2, \\ \gamma_4 &= \frac{4603055}{162} + \frac{135680}{27}\zeta(3) - 8800\zeta(5) \\ &\quad + \left( -\frac{91723}{27} - \frac{34192}{9}\zeta(3) + 880\zeta(4) + \frac{18400}{9}\zeta(5) \right) N_L \\ &\quad + \left( \frac{5242}{243} + \frac{800}{9}\zeta(3) - \frac{160}{3}\zeta(4) \right) N_L^2 \\ &\quad + \left( -\frac{332}{243} + \frac{64}{27}\zeta(3) \right) N_L^3, \end{aligned}$$

where  $N_L$  is the number of active light quark flavors at the scale  $\mu$ , i.e. flavors with masses  $\leq \mu$ , and  $\zeta$  is the Riemann zeta function ( $\zeta(3) \simeq 1.2020569$ ,  $\zeta(4) \simeq 1.0823232$ , and  $\zeta(5) \simeq 1.0369278$ ). Eq. (60.2) must be solved in conjunction with the RG equation for  $\overline{\alpha}_s(\mu)$  given in [1]. In addition, as the renormalization scale crosses quark mass thresholds one needs to match the scale dependence of  $\overline{m}$  below and above the threshold. There are finite threshold corrections; the necessary formulae can be found in Ref. [5].

### 60.3 Lattice QCD

*Ab initio* lattice QCD calculations of the fundamental parameters of QCD, i.e. the coupling constant and quark masses (except for the top-quark mass), has become a precision science (see also the sections on Lattice Quantum Chromodynamics and on Quantum Chromodynamics in this *Review*). Here we only briefly recall those features which are required for the determination of quark masses. For more details on lattice QCD, please see the section on Lattice Quantum Chromodynamics.

The inputs into lattice calculations are bare, dimensionless versions of the fundamental parameters of QCD: the bare strong coupling and the bare quark masses in lattice units, i.e. multiplied by the lattice spacing  $a$ , the distance between neighboring points of the lattice. The physical values of the quark masses are determined by computing as many physical quantities as there are

parameters, including one for the gauge coupling. In practice, the computations are performed for multiple lattice spacings, determined by the bare coupling via dimensional transmutation, and a number of bare quark masses. After renormalizing these masses and converting them to physical units, the resulting renormalized masses are interpolated to their physical values. This is achieved by requiring that the computed physical quantities reproduce their measured values as the  $a \rightarrow 0$  limit to our continuum world is taken. Such a tuning of fundamental parameters is required in any regularization of QCD: lattice QCD is simply a tool to perform QCD calculations in the low-energy, nonperturbative domain.

As discussed in the previous paragraph, one must renormalize the bare quark masses of lattice QCD simulations. The values of these bare masses refer to a particular discretization of QCD with the lattice spacing as the ultraviolet cut-off. In order for the resulting physical masses to be useful in phenomenological applications, it is necessary to give their values in some standard renormalization scheme such as  $\overline{\text{MS}}$ . Provided that both the ultraviolet cut-off  $a^{-1}$  and the renormalization scale  $\mu$  are much greater than  $\Lambda_{\text{QCD}}$ , the bare masses can be renormalized in perturbation theory. However, in order to avoid uncertainties due to the unknown higher-order coefficients in lattice perturbation theory, most results obtained recently use *nonperturbative renormalization*. This procedure relates the bare masses to those defined in renormalization schemes which can be realized directly in lattice and continuum QCD, e.g. those obtained from quark and gluon Green functions at specified momenta in the Landau gauge [6] or those defined using finite-volume techniques and the Schrödinger functional [7], but not  $\overline{\text{MS}}$  that is only defined for dimensional regularization. These methods require  $\mu \gg \Lambda_{\text{QCD}}$  so that unwanted (nonperturbative) corrections proportional to inverse powers of  $\mu$ , which appear in some approaches, remain small corrections that can be identified and removed. This condition is also necessary so that matching to other schemes can be performed reliably in perturbation theory. Moreover these methods require  $a^{-1} \gg \mu$  so that cutoff effects are small enough to be extrapolated away when the continuum limit is taken. The conversion to the  $\overline{\text{MS}}$  scheme is then performed using continuum perturbation theory, which is more readily obtained to higher orders and is usually better behaved than its lattice counterpart.

It is important to note that the final renormalized values for the quark masses must be the same in the continuum limit for any valid discretization of QCD and for any sensible choice of the physical quantities used for calibration, as long as the calculation is performed with a sufficient number of sea-quark flavors and with the relevant electromagnetic and strong-isospin breaking corrections for the claimed precision. It is also worth noting that issues surrounding the renormalization of quark masses disappear when considering pairwise ratios of these masses in QCD alone. Indeed, if the same renormalization scheme and scale are implemented, also for the definition of QCD in the absence or electromagnetism,<sup>1</sup> QCD renormalization factors are identical for all quark flavors, and these factors therefore cancel exactly in quark-mass ratio<sup>2</sup>. In particular, this means that these ratios are scheme and scale independent up to possible QED corrections. Moreover these ratios may suffer less from the uncertainties in the determination of the lattice scale, in particular in cases where the quantities used to fix the two quarks masses depend on these masses in a similar way, because these ratios are dimensionless. Thus, quark-mass ratios are often determined with significantly higher precision using lattice QCD than are the individual masses.

The determination of quark masses using lattice simulations is well established and the current emphasis is on the reduction and control of uncertainties. With better methods, improved algorithms and access to more powerful computing resources, the precision of the results has improved significantly in recent years. Vacuum polarization effects are included with  $N_f = 2, 2 + 1$  or  $N_f = 2 + 1 + 1$  flavors of sea quarks. The number 2 here indicates that the up and down quarks

<sup>1</sup>Since QCD without QED is not the real world, it makes a difference, for example, which hadron mass is used to tune a quark mass and the QED correction will differ for different choices.

<sup>2</sup>The same remains true in the presence of QED if the two quarks in the ratio have the same charge.

are degenerate. The first +1 corresponds to the inclusion of strange sea-quark effects and the second +1 to those of the charm. Simulations with 2 + 1 and 2 + 1 + 1 flavors represent controlled approximations to physical QCD at the low energies considered for quark mass determinations, up to corrections of  $O((\Lambda_{\text{QCD}}/m_c)^2/N_c)$  and  $O((\Lambda_{\text{QCD}}/m_b)^2/N_c)$ , respectively. This is not the case for simulations with  $N_f = 2$  or in which vacuum polarization effects are treated as a mean field (this is the so-called *quenched* approximation) and results obtained in those frameworks will not enter the discussion here.

Particularly pleasing is the observation that different formulations of lattice QCD and different approaches, with different systematic uncertainties, yield results which are largely consistent with each other. This gives us broad confidence in the estimates of the systematic errors. As the precision of the results approaches (or even exceeds in some cases) 1%, isospin breaking effects, including electromagnetic corrections need to be considered and this is beginning to be done as will be discussed below. In particular, a reliable estimate of these effects is required for determining the individual  $u$  and  $d$  quark masses.

Members of the lattice QCD community have organized a Flavour Lattice Averaging Group (FLAG) which, against stated quality criteria, critically reviews quantities computed in lattice QCD that are relevant to flavor physics and presents its view of the current status of the results. This includes the determination of quark masses. The latest edition reviews lattice results that appeared in print before April 30th 2021 [8]. Since that deadline and the writing of the previous edition of the present review, the only new results on quark masses that have been published are those of Ref. [9].

#### 60.4 Averaging procedure for lattice QCD results

In this review we present summary numbers for each quark mass which are solely based on averages of lattice results. Those summary numbers differ from those presented in the particle listings, for three reasons. The first is that they do not include non-lattice estimates of the quark masses. The second is that they may differ slightly in the lattice results retained, as made explicit here and in the listings. The third difference comes from the averaging procedures, which are described in the listings and below. In the listings, summary numbers are obtained from averages of the individual quark-mass estimates, weighted by their total errors. Moreover, the resulting one-standard-deviation errors are rescaled by the usual  $\sqrt{\chi^2/\text{dof}}$  factor, which is further multiplied by 1.645 to obtain the uncertainty corresponding to a 90% confidence interval. Nevertheless, the averages presented here and in the listings are fully consistent, with comparable uncertainties, signalling the robustness of the two combination procedure.

The results on which the lattice summary numbers presented here are those used in the FLAG reviews, supplemented by any new calculations that satisfy their selection criteria. The methodology that we use is also close to that of FLAG, but here we make an attempt to separate uncertainties into their statistical and systematic components, to provide some idea of what the relative weight of each is for different quark masses. In some cases, also, slightly different choices are also made. Thus, agreement between the results obtained is a further sign of the robustness of the combination procedures used here, as well as in FLAG.

We now lay out our averaging procedure. For each of  $m_{ud}$ ,  $m_s$  or  $m_s/m_{ud}$ , and for each of  $N_f = 2 + 1$  or  $N_f = 2 + 1 + 1$ , we perform averages of the relevant, individual lattice results with weights that include the sum in quadrature of the statistical and systematic uncertainties. The weights also account for correlations by assuming that they are 100% on any uncertainty components that may be correlated. This procedure yields a central value and a total error for each of the quantities of interest. The total uncertainty is rescaled by the usual factor of  $\sqrt{\chi^2/\text{dof}}$  if that factor is larger than 1, where  $\chi^2$  is the one corresponding to the weighted average procedure.

To isolate the statistical component of the uncertainty on any given average, like for the total uncertainty we use the standard error formula for correlated data, this time only taking into account the statistical uncertainties on the different collaborations' results. If the total error has been rescaled, this statistical uncertainty is multiplied by the same factor. The candidate systematic error on an average is then obtained by subtracting in quadrature the statistical uncertainty from the total one. For each quantity, the final systematic error on the average is the largest of two uncertainties: the candidate systematic error and the minimum of the systematic uncertainties on the individual lattice determinations. This guarantees that the final systematic error is no smaller than the one of any given calculation. We then repeat this whole procedure to average the  $N_f = 2 + 1$  and  $N_f = 2 + 1 + 1$  averages, for reasons discussed below. In cases where the error separation is not explicit in a particular collaboration's result, we assume that the quoted error is statistical and take, as the minimum of systematic error, the smallest non-vanishing one among those of the other individual results. While this procedure may change the balance between statistical and systematic errors, it does change the average (which is weighted by the total errors of the individual entries and the correlations between them) and does not decrease the total error. When quantities involving  $m_c$  or  $m_b$  are averaged, we consider only the averages of  $N_f = 2 + 1 + 1$  calculations, for reasons also explained below.

The values of the individual results used in the averages obtained below can be found in the quark properties listings of this *Review*.

## 60.5 Light quarks

In this section we review the determination of the masses of the light quarks  $u$ ,  $d$  and  $s$  from lattice simulations and then discuss the consequences of the approximate chiral symmetry.

### 60.5.1 Lattice QCD results

The most reliable determinations of the strange quark mass,  $m_s$ , and of the up and down quark masses,  $m_u$  and  $m_d$ , are obtained from lattice simulations. This is reflected in part by the uncertainties of individual determinations of these masses given in this *Review's* listings of quark properties: lattice results typically have smaller error bars than phenomenological extractions. As explained in the previous section, the simulations are generally performed in the isospin-symmetric limit of QCD, i.e. with degenerate up and down quarks ( $m_u = m_d$ ) and the electromagnetic coupling  $\alpha$  set to zero. Thus, it is initially  $m_{ud} = (m_u + m_d)/2$  which is obtained from the computations. The numerical values of these masses depend on the choice of renormalization scale and scheme.<sup>3</sup>

Below we discuss the derivation of  $m_u$  and  $m_d$  separately, but we start by briefly presenting our estimates for  $m_{ud}$ ,  $m_s$  and their ratio, using the latest lattice results. FLAG [8] bases its summary numbers on references that the authors consider to have the most reliable estimates of systematic uncertainties, i.e. [12–18] for  $N_f = 2 + 1$  and [19–24] for  $N_f = 2 + 1 + 1$  flavors of sea quarks. Note that the  $N_f = 2 + 1 + 1$  results of Ref. [9] did not make the FLAG deadline. In the estimates given below, we use the fact that Ref. [9] supersedes Ref. [22].

Applying the averaging procedure described in the previous section and the results of Refs. [12–18], for  $N_f = 2 + 1$  we obtain  $\overline{m}_{ud} = (3.382 \pm 0.026 \pm 0.040[\pm 0.050])$  MeV,  $\overline{m}_s = (92.24 \pm 0.24 \pm 1.00[\pm 1.03])$  MeV and  $(\overline{m}_s/\overline{m}_{ud}) = 27.42 \pm 0.05 \pm 0.12[\pm 0.12]$ . These numbers are  $\overline{m}_{ud} =$

<sup>3</sup> If QED is not included, then quark masses also depend on the conditions used to match isospin-symmetric QCD to the low energy effective theory of the Standard Model, which includes QCD and QED with  $m_u \neq m_d$ . Those matching conditions, which define a matching scheme, are quite similar in the calculations reported on below. Moreover, they have been designed to minimize the impact of isospin-breaking effects on isospin-symmetric combinations of quark masses, such  $m_{ud}$  and  $m_s$  here, and  $m_c$  and  $m_b$  below. Thus, as does FLAG [8], we generally assume that any remaining differences between schemes are within the quoted errors. Such an assumption is certainly true for  $\overline{m}_c(\overline{m}_c)$  [10], where QED effects have been studied explicitly. On the other hand, in the case of  $m_b/m_c$ , where the effects of QED have also been studied explicitly [11] and where uncertainties are less than 0.3%, these effects are no longer negligible.

$(3.421 \pm 0.035 \pm 0.049[\pm 0.060])$  MeV,  $\bar{m}_s = (92.93 \pm 0.51 \pm 0.37[\pm 0.063])$  MeV and  $(\bar{m}_s/\bar{m}_{ud}) = 27.25 \pm 0.03 \pm 0.07[\pm 0.08]$  using the latest  $N_f = 2 + 1 + 1$  calculations [9, 19–21, 23, 24]. The masses are given in the  $\overline{\text{MS}}$  scheme at a renormalization scale of 2 GeV. In these averages, the first error is statistical, the second systematic and the third one in brackets is the sum in quadrature of the first two. Also, QED corrections have been accounted for only to the extent that they have been removed from the hadron mass inputs phenomenologically.

A few comments are in order concerning the compatibility of the results which enter these averages. We begin with the case of  $N_f = 2 + 1$ . For  $m_{ud}$  the results show a good degree of compatibility with a  $\chi^2/\text{dof} = 5.7/4$  and the final systematic uncertainty is given by the one of Ref. [17]. The level of compatibility is similar in the case of  $m_s$  and the final systematic uncertainty is also given by the one of Ref. [17]. For  $m_s/m_{ud}$  the results are even more compatible and here the final systematic error is derived from the averaging procedure. In the case of  $N_f = 2 + 1 + 1$ , there are only two results for  $m_{ud}$  [9, 20] and their level of compatibility is poor, with  $\chi^2/\text{dof} = 6.4/1$ . This is due to the value of Ref. [9],  $m_{ud} = (3.636 \pm 0.066 \pm 0.059)$  MeV, which is significantly larger than all other determinations, including those obtained in  $N_f = 2 + 1$  determinations. The scenario repeats for  $m_s$ , but is diluted by the fact that there are four results [9, 19–21] which enter the average. However, compatibility is recovered for  $m_s/m_{ud}$ , suggesting a problem in the determination of the renormalization constants or the lattice spacing in at least one of the calculations. Since the different collaborations use different formulations of lattice QCD, the (relatively) small variations of the results between the groups provides important information about the reliability of the estimates. Furthermore, the conservative nature of our method for estimating uncertainties is confirmed by the excellent agreement of the averages obtained for  $N_f = 2 + 1$  and  $N_f = 2 + 1 + 1$  calculations.

Despite being reported in the  $\overline{\text{MS}}$  scheme at a renormalization scale of 2 GeV, the schemes in simulations with  $2 + 1$  and  $2 + 1 + 1$  sea-quark flavors actually differ. This is because the former are renormalized with  $N_L = 3$  and the latter with  $N_L = 4$ . Thus for a comparison one should convert the results to the same scheme. That is not the case for  $(m_s/m_{ud})$ , where renormalization factors cancel in the absence of electromagnetic corrections. The conversion of the  $N_f = 2 + 1$  results to the  $N_L = 4$  scheme can be performed, for instance, by running them down to the charm threshold in the  $N_L = 3$  theory, matching the results to the  $N_L = 4$  theory at that scale, and running them back up to 2 GeV with four active flavors. Such a conversion leads to an upward shift in the values of the quark masses of only around 0.3%, well within the quoted uncertainties on the quark masses themselves. Quark-mass ratios are not concerned by this upward shift, because it is identical for the numerator and denominator. There are also nonperturbative contributions to this procedure, of  $O((\Lambda_{\text{QCD}}/m_c)^2/N_c) \sim 2\%$ . However lattice QCD studies of these effects have shown that they are typically an order of magnitude smaller on hadronic quantities or on  $\Lambda_{\text{QCD}}$  [25, 26]. Given that the total errors on the  $N_f = 2 + 1$  results for  $\bar{m}_{ud}$  and  $\bar{m}_s$  are larger than 1%, we consider these matching effects negligible. In addition, for  $m_{ud}$  in particular there are six  $N_f = 2 + 1$  calculations in which all sources of uncertainty have been accounted for, while there are only two for  $N_f = 2 + 1 + 1$ . Thus, we average the averages from the two frameworks in the same way as above (see Sec. 60.4), yielding as a final lattice QCD estimate in the  $\overline{\text{MS}}$  scheme at  $\mu = 2$  GeV in the  $N_L = 4$  theory:

$$\bar{m}_{ud} = (3.397 \pm 0.021 \pm 0.040[\pm 0.045]) \text{ MeV}, \quad (60.4)$$

$$\bar{m}_s = (92.74 \pm 0.22 \pm 0.49[\pm 0.54]) \text{ MeV}, \quad (60.5)$$

and

$$r_s \equiv \frac{\bar{m}_s}{\bar{m}_{ud}} = 27.30 \pm 0.03 \pm 0.07[\pm 0.08]. \quad (60.6)$$

In these averages, the first error is statistical, the second systematic and the third one in brackets is the sum in quadrature of the first two. QED corrections have been accounted for only to the extent that they have been removed from the hadron mass inputs phenomenologically. It is worth noting that the result for  $r_s$  in Eq. (60.6) is identical to the ratio of  $m_s$  and  $m_{ud}$  from Eqs. (60.4) and (60.5), within the four digits for  $r_s$  given in Eq. (60.6), albeit with a significantly smaller total uncertainty.

To obtain the individual values of  $\overline{m}_u$  and  $\overline{m}_d$  requires the introduction of isospin breaking effects, including electromagnetism. This is now being done completely using lattice field theory, albeit neglecting electromagnetic effects in the sea in most cases (for an exception, see Ref. [27] which includes a calculation of the valence and sea QED contributions to  $\Delta M_K^2 = M_{K^0}^2 - M_{K^+}^2$  that is critical for determining  $\overline{m}_d - \overline{m}_u$ ). The effect of neglecting QED contributions from the sea on the  $u$  and  $d$  quark masses has been estimated in Ref. [28] to induce a contribution to the uncertainty that ranges from about 3% in  $\overline{m}_u/\overline{m}_d$  to less than 1% in  $\overline{m}_d$ . FLAG has also reviewed results for these masses [8]. Again, they separate results obtained from  $N_f = 2 + 1$  and  $N_f = 2 + 1 + 1$  simulations. For the former, they use the results of Ref. [28] and for the latter, those of Refs. [20, 29, 30].

To obtain our estimates, we proceed as for the light quark masses above. For  $N_f = 2 + 1$ , our estimates are the results of the only calculation in which all sources of uncertainty are accounted for [28]. These are  $\overline{m}_u = (2.27 \pm 0.06 \pm 0.06[\pm 0.09])$  MeV,  $\overline{m}_d = (4.67 \pm 0.06 \pm 0.06[\pm 0.09])$  MeV and  $\overline{m}_u/\overline{m}_d = (0.485 \pm 0.011 \pm 0.016[\pm 0.020])$ . For  $N_f = 2 + 1 + 1$ , they rely on Refs. [20, 29, 30]. Our estimates are  $\overline{m}_u = (2.14 \pm 0.04 \pm 0.07[\pm 0.08])$  MeV,  $\overline{m}_d = (4.70 \pm 0.03 \pm 0.04[\pm 0.05])$  MeV and  $\overline{m}_u/\overline{m}_d = (0.460 \pm 0.008 \pm 0.018[\pm 0.020])$ . These masses are given in the  $\overline{\text{MS}}$  scheme at 2 GeV in the  $N_L = 3$  and 4 theories, respectively. Given the size of the uncertainties on these averages, tiny corrections due to QED effects in the renormalization constants have been neglected.

In obtaining the  $N_f = 2 + 1 + 1$  averages, there is a significant  $\chi^2/1 = 4.7$  tension between the two results for  $m_u$  [20, 29] and a  $\chi^2/1 = 3.4$  one between those for  $m_u/m_d$  [29, 30]. These tensions are accounted for by the usual rescaling of uncertainties. However, agreement is found between the two results for  $m_d$  [20, 29]. Moreover, at the level of our conservative uncertainties, agreement is found with the  $N_f = 2 + 1$  results.

Again, given the small number of results and the few percent uncertainties, we average the  $N_f = 2 + 1$  and  $N_f = 2 + 1 + 1$  results, as explained in Sec. 60.4, to obtain our final estimates:

$$\overline{m}_u = (2.20 \pm 0.04 \pm 0.06[\pm 0.07]) \text{ MeV}, \quad (60.7)$$

$$\overline{m}_d = (4.69 \pm 0.03 \pm 0.04[\pm 0.05]) \text{ MeV}, \quad (60.8)$$

$$r \equiv \frac{\overline{m}_u}{\overline{m}_d} = (0.473 \pm 0.007 \pm 0.016[\pm 0.017]), \quad (60.9)$$

where the three uncertainties have the same meaning as described after Eq. 60.6.  $r$  in Eq. (60.9) is the combination of direct lattice computations of that ratio in which some systematics cancel. The averages are given in the  $\overline{\text{MS}}$  scheme at 2 GeV in the  $N_L = 4$  theory and, as above, tiny corrections due to QED effects in the renormalization constants have been neglected. Within one standard deviation it agrees with the ratio of  $\overline{m}_u$  over  $\overline{m}_d$  from Eqs. (60.7) and (60.8).

It is also worth noting that  $m_u$  differs from zero by more than 30 times the quoted error, making a scenario in which  $m_u = 0$  very unlikely. This is important because there would be no strong CP problem if  $m_u$  were to vanish.

### 60.5.2 Chiral Perturbation Theory

For light quarks, one can use the techniques of chiral perturbation theory [31–33] to extract quark mass ratios. The mass term for light quarks in the QCD Lagrangian is

$$\bar{\Psi}M\Psi = \bar{\Psi}_L M \Psi_R + \bar{\Psi}_R M^\dagger \Psi_L, \quad (60.10)$$

where  $M$  is the light quark mass matrix,

$$M = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}, \quad (60.11)$$

$\Psi = (u, d, s)$ , and  $L$  and  $R$  are the left- and right-chiral components of  $\Psi$  given by  $\Psi_{L,R} = P_{L,R}\Psi$ ,  $P_L = (1-\gamma_5)/2$ ,  $P_R = (1+\gamma_5)/2$ . The mass term is the only term in the QCD Lagrangian that mixes left- and right-handed quarks. In the limit  $M \rightarrow 0$ , there is an independent  $SU(3) \times U(1)$  flavor symmetry for the left- and right-handed quarks. The vector  $U(1)$  symmetry is baryon number; the axial  $U(1)$  symmetry of the classical theory is broken in the quantum theory due to the anomaly. The remaining  $G_\chi = SU(3)_L \times SU(3)_R$  chiral symmetry of the QCD Lagrangian is spontaneously broken to  $SU(3)_V$ , which, in the limit  $M \rightarrow 0$ , leads to eight massless Goldstone bosons, the  $\pi$ 's,  $K$ 's, and  $\eta$ .

The symmetry  $G_\chi$  is only an approximate symmetry, since it is explicitly broken by the quark mass matrix  $M$ . The Goldstone bosons acquire masses which can be computed in a systematic expansion in  $M$ , in terms of low-energy constants, which are unknown nonperturbative parameters of the effective theory, and are not fixed by the symmetries. One treats the quark mass matrix  $M$  as a uniform, external field that transforms under  $G_\chi$  as  $M \rightarrow LMR^\dagger$ , where  $\Psi_L \rightarrow L\Psi_L$  and  $\Psi_R \rightarrow R\Psi_R$  are the  $SU(3)_L$  and  $SU(3)_R$  transformations, and writes down the most general Lagrangian invariant under  $G_\chi$ . Then one sets  $M$  to its given constant value Eq. (60.11), which implements the symmetry breaking. To first order in  $M$  one finds that [34]

$$\begin{aligned} m_{\pi^0}^2 &= B(m_u + m_d), \\ m_{\pi^\pm}^2 &= B(m_u + m_d) + \Delta_{\text{em}}, \\ m_{K^0}^2 &= m_{\bar{K}^0}^2 = B(m_d + m_s), \\ m_{K^\pm}^2 &= B(m_u + m_s) + \Delta_{\text{em}}, \\ m_\eta^2 &= \frac{1}{3}B(m_u + m_d + 4m_s), \end{aligned} \quad (60.12)$$

with two unknown constants  $B$ , related to the light quark condensate, and  $\Delta_{\text{em}}$ , the electromagnetic mass difference. From Eq. (60.12), one can determine the quark mass ratios [34]

$$\begin{aligned} \frac{m_u}{m_d} &= \frac{2m_{\pi^0}^2 - m_{\pi^+}^2 + m_{K^+}^2 - m_{K^0}^2}{m_{K^0}^2 - m_{K^+}^2 + m_{\pi^+}^2} = 0.56, \\ \frac{m_s}{m_d} &= \frac{m_{K^0}^2 + m_{K^+}^2 - m_{\pi^+}^2}{m_{K^0}^2 + m_{\pi^+}^2 - m_{K^+}^2} = 20.2, \end{aligned} \quad (60.13)$$

to lowest order in chiral perturbation theory, with an error which will be estimated below. Since the mass ratios extracted using chiral perturbation theory use the symmetry transformation property of  $M$  under the chiral symmetry  $G_\chi$ , it is important to use a renormalization scheme for QCD that does not change this transformation law. Any mass-independent subtraction scheme such as  $\overline{\text{MS}}$  is

suitable. The ratios of quark masses are scale independent in such a scheme (up to electromagnetic corrections), and Eq. (60.13) can be taken to be the ratio of  $\overline{\text{MS}}$  masses. Chiral perturbation theory cannot determine the overall scale of the quark masses, since it uses only the symmetry properties of  $M$ , and any multiple of  $M$  has the same  $G_\chi$  transformation law as  $M$ .

Chiral perturbation theory is a systematic expansion in powers of the light quark masses. The typical expansion parameter is  $m_K^2/\Lambda_\chi^2 \sim 0.25$  if one uses SU(3) chiral symmetry, and  $m_\pi^2/\Lambda_\chi^2 \sim 0.02$  if instead one uses SU(2) chiral symmetry. Electromagnetic effects at the few percent level also break SU(2) and SU(3) symmetry. The mass formulæ Eq. (60.12) were derived using SU(3) chiral symmetry, and are expected to have approximately a 25% uncertainty due to second order corrections. This estimate of the uncertainty yields results consistent with the lattice results summarized in Eq. (60.5)–(60.6).

There is a subtlety which arises when one tries to determine quark mass ratios at second order in chiral perturbation theory. The second order quark mass term [35]

$$(M^\dagger)^{-1} \det M^\dagger \quad (60.14)$$

(which can be generated by instantons) transforms in the same way under  $G_\chi$  as  $M$ . Chiral perturbation theory cannot distinguish between  $M$  and  $(M^\dagger)^{-1} \det M^\dagger$ ; one can make the replacement  $M \rightarrow M(\lambda) = M + \lambda M(M^\dagger M)^{-1} \det M^\dagger$  in the chiral Lagrangian,

$$\begin{aligned} M(\lambda) &= \text{diag}(m_u(\lambda), m_d(\lambda), m_s(\lambda)) \\ &= \text{diag}(m_u + \lambda m_d m_s, m_d + \lambda m_u m_s, m_s + \lambda m_u m_d), \end{aligned} \quad (60.15)$$

and leave all observables unchanged.

The combination

$$\left(\frac{m_u}{m_d}\right)^2 + \frac{1}{Q^2} \left(\frac{m_s}{m_d}\right)^2 = 1 \quad (60.16)$$

where

$$Q^2 = \frac{m_s^2 - m_{ud}^2}{m_d^2 - m_u^2}, \quad m_{ud} = \frac{1}{2}(m_u + m_d),$$

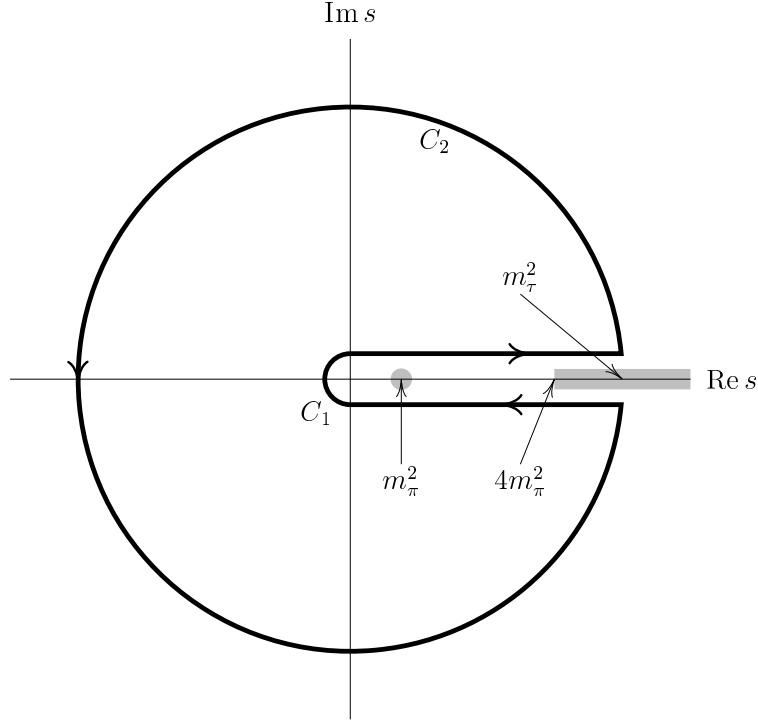
is insensitive to the transformation in Eq. (60.15). Eq. (60.16) gives an ellipse in the  $m_u/m_d$ – $m_s/m_d$  plane. The ellipse is well-determined by chiral perturbation theory, but the location on the ellipse, and the absolute normalization of the quark masses, have larger uncertainties.

A leading-order result for  $Q$  in SU(3) chiral perturbation theory can be derived using Eq. (60.12) and the values for the relevant meson masses. This result actually holds to next-to-leading order, yielding  $Q \stackrel{\text{NLO}}{=} 24.3$ . Phenomenologically, the preferred way to determine  $Q$  is from  $\eta \rightarrow 3\pi$  decay, giving the smaller value  $Q = 22.1(7)$  [36]. Lattice QCD collaborations have also reported determinations of  $Q$ . Using  $N_f = 2 + 1$  simulations, Ref. [28] obtains  $Q = 23.4 \pm 0.6$  and Ref. [29] determines  $Q = 23.8 \pm 1.1$  with  $N_f = 2 + 1 + 1$  simulations, results which are fully compatible. These results are also compatible with  $Q = 23.4 \pm 0.5$ , obtained from  $Q^2 = (r_s^2 - 1)(1 + r)/[4(1 - r)]$  and from our lattice averages for  $r_s$  and  $r$  from Eqs. (60.6) and (60.9), respectively. Given the size of the uncertainties on the lattice results for  $Q$ , it is safe to neglect effects due to the definition of isosymmetric QCD (see footnote 3). On the whole, the lattice results are larger than the one from phenomenology [36]. This difference could be due to surprisingly large corrections to the NLO substitution of meson masses by  $Q$  in the phenomenological determination, as suggested by the authors of [36].

The absolute normalization of the quark masses cannot be determined using chiral perturbation theory. Other methods, such as lattice simulations discussed above, or spectral function sum rules [37, 38] for hadronic correlation functions reviewed next, are necessary.

### 60.5.3 Sum rules

Sum rule methods have been used extensively to determine quark masses and for illustration we briefly discuss here their application to hadronic  $\tau$  decays [39]. Other applications involve very similar techniques.



**Figure 60.1:** The analytic structure of  $\Pi(s)$  in the complex  $s$ -plane. The contours  $C_1$  and  $C_2$  are the integration contours discussed in the text, and the integral over the closed contour  $C_1 + C_2$  vanishes.  $m_\tau^2$  has not been drawn to scale;  $m_\tau^2 \sim 40(4m_\pi^2)$ .

The experimentally measured quantity is  $R_\tau$ ,

$$\frac{dR_\tau}{ds} = \frac{d\Gamma/ds(\tau^- \rightarrow \text{hadrons} + \nu_\tau(\gamma))}{\Gamma(\tau^- \rightarrow e^- \bar{\nu}_e \nu_\tau(\gamma))} \quad (60.17)$$

the hadronic invariant mass spectrum in semihadronic  $\tau$  decay, normalized to the leptonic  $\tau$  decay rate. It is useful to define  $q$  as the total momentum of the hadronic final state, so  $s = q^2$  is the hadronic invariant mass. The total hadronic  $\tau$  decay rate  $R_\tau$  is then given by integrating  $dR_\tau/ds$  over the kinematically allowed range  $0 \leq s \leq M_\tau^2$ .

$R_\tau$  can be written as

$$R_\tau = 12\pi \int_0^{M_\tau^2} \frac{ds}{M_\tau^2} \left(1 - \frac{s}{M_\tau^2}\right)^2 \left[ \left(1 + 2\frac{s}{M_\tau^2}\right) \text{Im} \Pi^T(s) + \text{Im} \Pi^L(s) \right] \quad (60.18)$$

where the hadronic spectral functions  $\Pi^{L,T}$  are defined from the time-ordered correlation function of two weak currents ( $j^\mu(x)$  and  $j^\nu(0)$ ) by

$$\Pi^{\mu\nu}(q) = i \int d^4x e^{iq \cdot x} \langle 0 | T(j^\mu(x) j^\nu(0)^\dagger) | 0 \rangle, \quad (60.19)$$

$$\Pi^{\mu\nu}(q) = (-g^{\mu\nu} + q^\mu q^\nu) \Pi^T(s) + q^\mu q^\nu \Pi^L(s), \quad (60.20)$$

and the decomposition Eq. (60.19) is the most general possible structure consistent with Lorentz invariance.

By the optical theorem, the imaginary part of  $\Pi^{\mu\nu}$  is proportional to the total cross-section for the current to produce all possible states. A detailed analysis including the phase space factors leads to Eq. (60.18). The spectral functions  $\Pi^{L,T}(s)$  are analytic in the complex  $s$  plane, with singularities along the real axis. There is an isolated pole at  $s = m_\pi^2$ , and single- and multi-particle singularities for  $s \geq 4m_\pi^2$ , the two-particle threshold. The discontinuity along the real axis is  $\Pi^{L,T}(s + i0^+) - \Pi^{L,T}(s - i0^+) = 2i\text{Im } \Pi^{L,T}(s)$ . As a result, Eq. (60.18) can be rewritten with the replacement  $\text{Im } \Pi^{L,T}(s) \rightarrow -i\Pi^{L,T}(s)/2$ , and the integration being over the contour  $C_1$ . Finally, the contour  $C_1$  can be deformed to  $-C_2$  without crossing any singularities, and so leaving the integral unchanged, i.e. the integral over the closed contour  $C_1 + C_2$  vanishes. One can derive a series of sum rules analogous to Eq. (60.18) by weighting the differential  $\tau$  hadronic decay rate by different powers of the hadronic invariant mass [40],

$$R_\tau^{kl} = \int_0^{M_\tau^2} ds \left(1 - \frac{s}{M_\tau^2}\right)^k \left(\frac{s}{M_\tau^2}\right)^l \frac{dR_\tau}{ds}. \quad (60.21)$$

This leads to the final form of the sum rule(s),

$$R_\tau^{kl} = -6\pi i \int_{C_2} \frac{ds}{M_\tau^2} \left(1 - \frac{s}{M_\tau^2}\right)^{2+k} \left(\frac{s}{M_\tau^2}\right)^l \left[\left(1 + 2\frac{s}{M_\tau^2}\right) \Pi^T(s) + \Pi^L(s)\right]. \quad (60.22)$$

The manipulations so far are completely rigorous and exact, relying only on the general analytic structure of quantum field theory. The left-hand side of the sum rule Eq. (60.22) is obtained from experiment. The right hand-side can be computed for  $s$  far away from any physical cuts using the operator product expansion (OPE) for the time-ordered product of currents in Eq. (60.18), and QCD perturbation theory. The OPE is an expansion in a series of local operators, and is an expansion about the  $q \rightarrow \infty$  limit. It gives  $\Pi^{L,T}(s)$  as an expansion in powers of  $\alpha_s(s)$  and  $\Lambda_{\text{QCD}}^2/s$ , and is valid when  $s$  is far (in units of  $\Lambda_{\text{QCD}}^2$ ) from any singularities in the complex  $s$ -plane.

The OPE gives  $\Pi^{L,T}(s)$  as a series in  $\alpha_s$ , quark masses, and various nonperturbative vacuum matrix elements. By computing  $\Pi^{L,T}(s)$  theoretically, and comparing with the experimental values of  $R_\tau^{kl}$ , one determines various parameters such as  $\alpha_s$  and the quark masses. The theoretical uncertainties in using Eq. (60.22) arise from neglected higher order corrections (both perturbative and nonperturbative), and because the OPE is no longer valid near the real axis, where  $\Pi^{L,T}$  have singularities. The contribution of neglected higher order corrections can be estimated as for any other perturbative computation. The error due to the failure of the OPE is more difficult to estimate. In Eq. (60.22), the OPE fails on the endpoints of  $C_2$  that touch the real axis at  $s = M_\tau^2$ . The weight factor  $(1 - s/M_\tau^2)$  in Eq. (60.22) vanishes at this point, so the importance of the endpoint can be reduced by choosing larger values of  $k$ .

Light quark masses are often determined using QCD sum rules [38], which are similar to the  $\tau$  sum rules. One takes the correlator of two light-quark-bilinear operators (e.g. an axial vector current), as in Eq. (60.18), and computes their Laplace transforms or moments

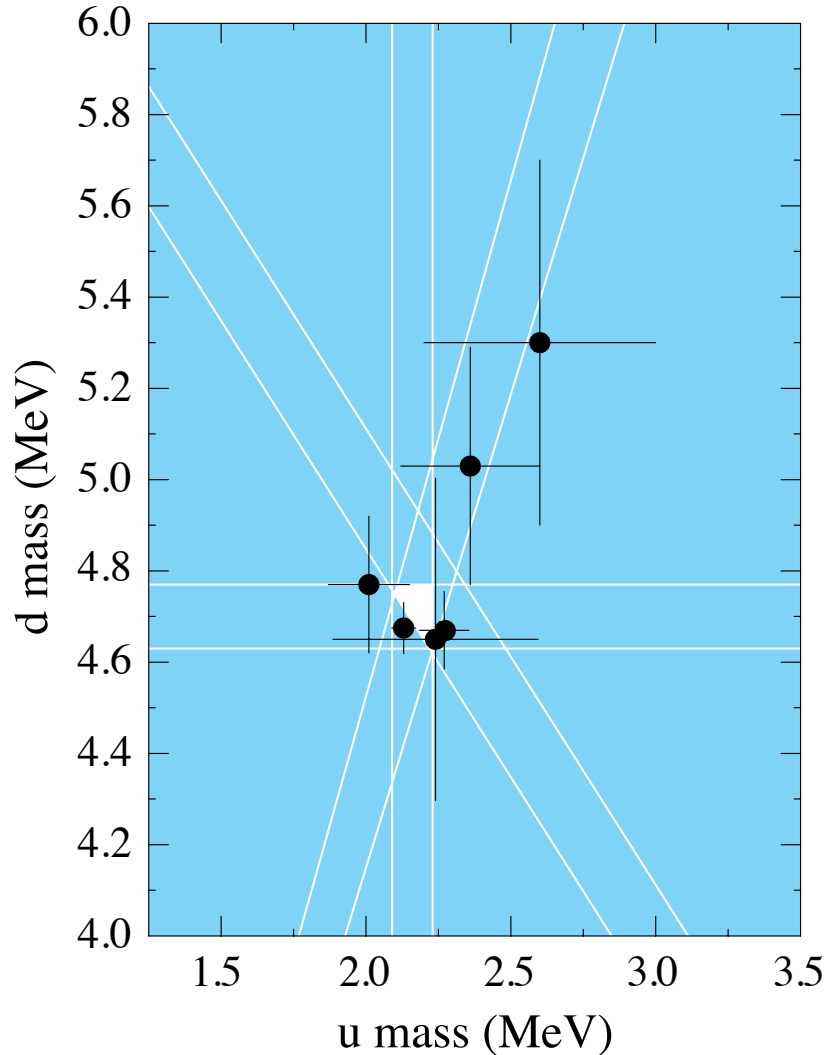
$$\mathcal{L}_n(\tau) = \int_0^\infty ds s^n e^{-\tau s} \text{Im } \Pi(s), \quad \mathcal{M}_n(Q^2) = \int_0^\infty \frac{ds}{(s + Q^2)^n} \text{Im } \Pi(s)$$

to get Laplace or moment sum rules, respectively. The quark masses are extracted by comparing the theoretical and experimental values of  $\mathcal{L}_n(\tau)$  and  $\mathcal{M}_n(Q^2)$ . Considerable theoretical effort has gone into optimizing  $n$  and  $Q^2$  to improve the precision of the resulting light quark masses.

## 60.6 Heavy quarks

### 60.6.1 Continuum approaches and results

For heavy quark physics one can exploit the fact that  $m_Q \gg \Lambda_{\text{QCD}}$  to construct effective theories ( $m_Q$  is the mass of the heavy quark  $Q$ ). The masses and decay rates of hadrons containing a single heavy quark, such as the  $B$  and  $D$  mesons can be determined using the heavy quark effective theory (HQET) [41]. The theoretical calculations involve radiative corrections computed in perturbation theory with an expansion in  $\alpha_s(m_Q)$  and nonperturbative corrections with an expansion in powers of  $\Lambda_{\text{QCD}}/m_Q$ . Due to the asymptotic nature of the QCD perturbation series, the two kinds of corrections are intimately related; an example of this are renormalon effects in the perturbative expansion which are associated with nonperturbative corrections.



**Figure 60.2:** The allowed region (shown in white) for up quark and down quark masses renormalized in the  $\overline{\text{MS}}$  scheme at 2 GeV, determined from the allowed ranges of mass parameters (see Fig. 60.3). The parameter  $(m_u + m_d)/2$  yields the two downward-sloping white lines, while  $m_u/m_d$  yields the two rising white lines originating at (0,0). The white region is the remaining allowed parameter space after applying the limits on  $m_u$ ,  $m_d$ ,  $(m_u + m_d)/2$  and  $m_u/m_d$ , where those limits are 90% confidence levels including the scale factors shown in the ideograms in the Data Listings.

Systems containing two heavy quarks such as the  $\mathcal{T}$  or  $J/\Psi$  are treated using non-relativistic QCD (NRQCD) [42]. The typical momentum and energy transfers in these systems are  $\alpha_s m_Q$ , and  $\alpha_s^2 m_Q$ , respectively, so these bound states are sensitive to scales much smaller than  $m_Q$ . However, smeared observables, such as the cross-section for  $e^+e^- \rightarrow \bar{b}b$  averaged over some range of  $s$  that includes several bound state energy levels, are better behaved and only sensitive to scales near  $m_Q$ . For this reason, most determinations of the  $c, b$  quark masses using perturbative calculations compare smeared observables with experiment [43–45]. The method is similar to that outlined for  $\tau$  decays. The current correlator in Eq. (60.18) is the electromagnetic current, and the experimental data is the value of  $R(s)$  in the threshold region for  $e^+e^- \rightarrow Q\bar{Q}$ . The theoretical values for the moments are computed using renormalization group improved calculations in non-relativistic QCD.

There are many continuum extractions of the  $c$  and  $b$  quark masses, some with quoted errors of 10 MeV or smaller. There are systematic effects of comparable size, which are typically not included in these error estimates. Reference [46], for example, shows that even though the error estimate of  $m_c$  using the rapid convergence of the  $\alpha_s$  perturbation series is only a few MeV, the central value of  $m_c$  can differ by a much larger amount depending on which algorithm (all of which are formally equally good) is used to determine  $m_c$  from the data. This leads to a systematic error from perturbation theory of around 20 MeV for the  $c$  quark and 25 MeV for the  $b$  quark. Electromagnetic effects, which also are important at this precision, are often not included. For this reason, we inflate the errors on the continuum extractions of  $m_c$  and  $m_b$ . The average values of  $m_c$  and  $m_b$  from continuum determinations are (see Sec. 60.7 for the 1S scheme)

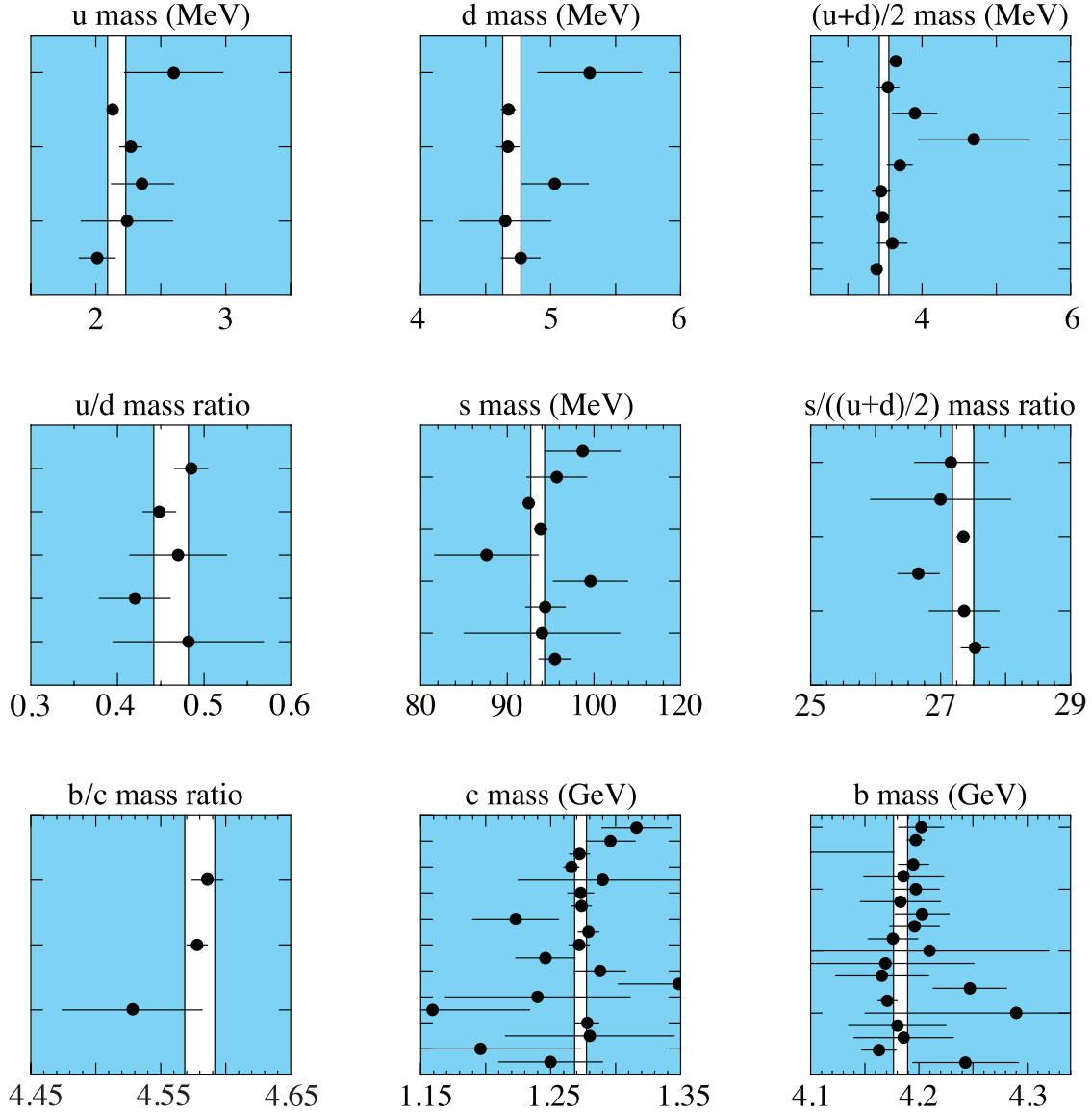
$$\overline{m}_c(\overline{m}_c) = (1.280 \pm 0.025)\text{GeV},$$

$$\overline{m}_b(\overline{m}_b) = (4.18 \pm 0.03)\text{GeV}, \quad m_b^{1\text{S}} = (4.65 \pm 0.03)\text{GeV}.$$

### 60.6.2 Lattice approaches and results

Lattice QCD simulations of a heavy quark  $Q$ , described in a relativistic fermion formulation, lead to potentially large discretization errors which are powers of  $am_Q$  (modulated by logarithms); the leading power depends on the formulation of lattice QCD being used and in most cases is quadratic. Clearly those errors can be reduced by performing simulations at smaller lattice spacings  $a$ , but also by using *improved* discretizations of the theory, in which the leading discretization errors are systematically eliminated. Recently, with more powerful computing resources, better algorithms and improved discretizations of QCD, it has become possible to perform simulations with quarks more massive than the charm and even up to the  $b$  [11], decreasing or eliminating the extrapolation which has to be performed to reach the  $m_b$ . It is worth noting that computations using a relativistic fermion formulation for the valence charm with  $N_f = 2 + 1$  sea-quark flavors, or with  $N_f = 2 + 1 + 1$  flavors and a relativistic  $b$ , suffer from partial-quenching effects. This is because the heavy valence quarks are absent from the sea. While these effects are not expected to be large in  $N_f = 2 + 1 + 1$  computations, they should be investigated when quoting results below the percent level in the case of  $N_f = 2 + 1$  sea-quark flavors. For that reason we will omit heavy-quark results obtained with  $N_f = 2 + 1$  sea-quark flavors from our averages and, more generally focus on  $N_f = 2 + 1 + 1$  results. Note that these partial-quenching effects are accounted for systematically in lattice calculations in which the heavy valence quarks are described by HQET or NRQCD.

Traditionally the charm quark mass is obtained by tuning its bare, simulation input value to reproduce the physical mass of charmonium mesons or of the  $D, D_s$  mesons (requiring a more precise tuning of the light quark masses and the computation of larger QED effects in some cases). This mass can then be renormalized and matched to the  $\overline{\text{MS}}$  scheme using the methods discussed for the light quarks.



**Figure 60.3:** The allowed regions (shown in white) for quark masses renormalized in the  $\overline{\text{MS}}$  scheme at 2 GeV. The points (taken from the Data Listings) are in chronological order with the more recent measurements shown at the top. The white regions indicate values allowed at the 90% confidence level including the scale factors shown in the ideograms in the Data Listings.

An alternative approach for obtaining the  $\overline{\text{MS}}$  mass was proposed in [47]. Euclidean-time moments of pseudoscalar, two-point functions of  $c\bar{c}$  quark-bilinear operators can readily be computed on the lattice and extrapolated to the continuum limit where they can be compared to perturbative calculations of the same quantities at 4-loop order. In this way, both the strong coupling constant and the charm quark mass can be determined with remarkably small errors. As this approach uses the same perturbative expressions for two-point correlators as the continuum determinations discussed in Sec. 60.6.1, it suffers from similar perturbation-theory, systematic errors. FLAG [8] has reviewed lattice determinations of the charm-quark mass obtained using both approaches. For their  $N_f = 2 + 1$  average they retain the results [13, 48–50], with [51] being published after their

deadline. As stated above, we focus here on  $N_f = 2 + 1 + 1$  results. For these, FLAG uses the results of Refs. [10, 19, 20, 22, 52].

Since the FLAG review, Refs. [22, 52] have been superseded by Ref. [9]. In that reference, only  $\overline{m}_c(3 \text{ GeV})$  is given and we multiply it by the running factor  $0.7739 \pm 0.0060$  from  $\mu = 3 \text{ GeV}$  to  $\overline{m}_c$  [8]. With this substitution and using the combination method described in Sec. 60.4, we obtain:

$$\overline{m}_c(\overline{m}_c) = (1.275 \pm 0.004 \pm 0.008[\pm 0.009]) \text{ GeV} \quad (60.23)$$

where the three uncertainties have the same meaning as described after Eq. (60.6). This result is in good agreement with the continuum result given above, but with a smaller error.

It is worth noting that three [10, 19, 20] of the four results entering this average agree, while the other [9] is over 2 standard deviations larger than the average. This is taken into account by stretching errors in the usual way, but the total uncertainty on the lattice average of  $\overline{m}_c$  will not significantly be reduced until this discrepancy is resolved. It should also be noted that [10] includes QED directly in the lattice calculation, albeit only for valence quarks, while some of the other computations account for these effects using phenomenology. For  $\overline{m}_c(\overline{m}_c)$  considered above, these effects are below the permil [10] and are therefore neglected here.

Historically, the main approach to controlling the discretization errors in lattice studies of  $b$ -quark physics was to perform simulations of effective theories such as HQET and NRQCD. This remains an important technique, both in its own right and in providing additional information for extrapolations from lower masses to the bottom region. Using effective theories,  $m_b$  is obtained from what is essentially a computation of the difference of  $M_{H_b} - m_b$ , where  $M_{H_b}$  is the mass of a hadron  $H_b$  containing a  $b$  quark. The relative error on  $m_b$  is therefore much smaller than that for  $M_{H_b} - m_b$ . The principal systematic errors are the matching of the effective theories to QCD and the presence of power divergences in  $a^{-1}$  in the  $1/m_b$  corrections which have to be subtracted numerically. A procedure for performing these subtractions fully nonperturbatively was proposed and implemented for the first time in [53].

The most recent  $N_f = 2 + 1 + 1$  lattice QCD determinations of the  $b$  quark mass rely on a variety of approaches, including Euclidean-time moments of correlation functions with [54] or without NRQCD [19] and interpolations [55, 56] (using results from HQET simulations) or extrapolations [11, 20] from above the charm mass to that of the  $b$ . One calculation [11] even includes simulations performed with a heavy quark whose mass is very near that of the  $b$  quark. The overall agreement of the results obtained using these very different approaches, which have different systematic errors, is a confirmation that the various groups control these uncertainties. As the range of heavy-quark masses which can be used in numerical simulations increases, the extrapolation or interpolation of results to the physical  $b$  mass, possibly including some obtained directly at that mass, are becoming ever more reliable (see e.g. Refs. [11, 20]).

Using the results from Refs. [11, 19, 20, 55, 56], converting those given in the  $N_L = 4$  theory to the  $N_L = 5$  one and applying the averaging method described in Sec. 60.4, we obtained the estimate:

$$\overline{m}_b(\overline{m}_b) = (4.196 \pm 0.009 \pm 0.009[\pm 0.012]) \text{ GeV} , \quad (60.24)$$

where the three uncertainties have the same meaning as in the estimates given above for light-quark masses.

Among the results entering this average, only one [11] accounts for electromagnetic effects. Using Refs. [10, 11], it is straightforward to show that QED effects on the value of  $\overline{m}_b$  at  $\mu = 3 \text{ GeV}$ , and thus at  $\mu = \overline{m}_b$ , are smaller than one permil in the separation scheme of those two references. Assuming that the same is true for the other results, we neglect these corrections here.

The lattice result of Eq. (60.24) is compatible with the average value of continuum results, but with a significantly smaller uncertainty. Note that FLAG [8] also provides an average of  $N_f = 2 + 1$  results, taken from [13, 50]. It is about one combined, standard deviations lower than the result quoted above.

As explained in Sec. 60.3, ratios of quark masses can have significantly smaller errors than the individual masses if they are determined within the same lattice QCD framework. This led HPQCD to leverage their precise determination of  $m_c$  [47] to determine  $m_s$  and  $m_{ud}$  [57], through a precise computation of  $m_c/m_s$  [57] and of  $m_s/m_{ud}$  [58]. The  $N_f = 2 + 1$  calculation [57] was updated using  $N_f = 2 + 1 + 1$  simulations in [19]. The ratio  $m_c/m_s$  was also computed in Refs [48, 59] with  $N_f = 2 + 1$  simulations and in Refs. [20, 22] with  $N_f = 2 + 1 + 1$  ones. On the basis of those references, FLAG [8] gives average results from  $N_f = 2 + 1$  and  $N_f = 2 + 1 + 1$  simulations. Here we focus on  $N_f = 2 + 1 + 1$  results and we replace the results of Ref. [22] by the collaboration's update [9]. Applying to the modern  $N_f = 2 + 1 + 1$  computations of Refs. [9, 19, 20] the combination procedure described in Sec. 60.4, we obtain the following lattice average:

$$\overline{m}_c/\overline{m}_s = 11.769 \pm 0.021 \pm 0.028[\pm 0.035],$$

where the three uncertainties have the same meaning as described after Eq. 60.6. The average is dominated by Ref. [20] that has a substantially smaller error than the other results. The result of Ref. [9] is a little more than 1 standard deviation below that average and the one of Ref. [19] close to  $2\sigma$  above, for a total  $\chi^2/\text{dof} = 5.0/2$  and a rescaling factor of  $\sim 1.6$  in the statistical and systematic errors.

The above ratio is renormalization scheme and scale independent, in the absence of electromagnetic corrections. None of the calculations above estimates the size of those corrections. Nevertheless, they are expected to still be smaller than the total uncertainty quoted above in a standard scheme for defining QCD in the isospin limit.

The ratio  $m_b/m_c$  has also been computed on the lattice. The two  $N_f = 2 + 1$  results which satisfy FLAG's criteria are [13, 50]. As explained above, we focus on  $N_f = 2 + 1 + 1$  calculations [11, 20, 55]. Ref. [11] is the only calculation which accounts for QED directly in the lattice simulations, albeit only for valence quarks, and thus must specify the renormalization scheme and scale of its result, which it chooses to be  $\overline{\text{MS}}$  at 3 GeV. Here we correct the results of Refs. [20, 55] by the QED factor 1.0017 obtained in Ref. [11], and ascribe to it a 100% systematic uncertainty with 100% correlation. This means that we assume that the differences due to the separation schemes used in each calculation lies within that error. Averaging the resulting numbers, as prescribed in Sec. 60.4, yields

$$\overline{m}_b/\overline{m}_c = 4.584 \pm 0.006 \pm 0.010[\pm 0.012],$$

where exceptionally the bar indicates that the masses are renormalized in the  $\overline{\text{MS}}$  at 3 GeV in the four-flavor theory. Here a scale factor of  $\sqrt{\chi^2/\text{dof}} = 1.30$  has been applied to the error bar. Indeed, [55] contributes 3.3 to the total  $\chi^2$ .

### 60.6.3 Warnings concerning the use of the pole mass

For an observable particle such as the electron, the position of the pole in the propagator is the definition of its mass. In QCD this definition of the quark mass is known as the pole mass. It is known that the on-shell quark propagator has no infrared divergences in perturbation theory [60, 61], so this provides a perturbative definition of the quark mass. However, the pole mass cannot be used to arbitrarily high accuracy because of nonperturbative infrared effects in QCD. In fact the full quark propagator has no pole because the quarks are confined, so that the pole mass cannot be defined outside of perturbation theory. The relation between the pole mass  $m_Q$  and the  $\overline{\text{MS}}$  mass

$\bar{m}_Q$ , used throughout this review, is known to three loops [62–65]

$$\begin{aligned}
m_Q = \bar{m}_Q(\bar{m}_Q) & \left\{ 1 + \frac{4\bar{\alpha}_s(\bar{m}_Q)}{3\pi} \right. \\
& + \left[ -1.0414 \sum_q \left( 1 - \frac{4}{3} \frac{\bar{m}_q}{\bar{m}_Q} \right) + 13.4434 \right] \left[ \frac{\bar{\alpha}_s(\bar{m}_Q)}{\pi} \right]^2 \\
& \left. + \left[ 0.6527N_L^2 - 26.655N_L + 190.595 \right] \left[ \frac{\bar{\alpha}_s(\bar{m}_Q)}{\pi} \right]^3 \right\}, \tag{60.25}
\end{aligned}$$

where  $\bar{\alpha}_s(\mu)$  is the strong interaction coupling constants in the  $\overline{\text{MS}}$  scheme, and the sum over  $q$  extends over the  $N_L$  flavors lighter than  $Q$ . The complete mass dependence of the  $\alpha_s^2$  term can be found in [62]; the mass dependence of the  $\alpha_s^3$  term is not known. For the  $b$ -quark, Eq. (60.25) reads

$$m_b = \bar{m}_b(\bar{m}_b) [1 + 0.10 + 0.05 + 0.03], \tag{60.26}$$

where the contributions from the different orders in  $\alpha_s$  are shown explicitly. The two and three loop corrections are comparable in size and have the same sign as the one loop term. This is a signal of the asymptotic nature of the perturbation series (there is a renormalon in the pole mass [66]). Such a badly behaved perturbation expansion can be avoided by directly extracting, from data, the mass defined in the  $\overline{\text{MS}}$  (used in this review) or other short-distance schemes (see below), without invoking the pole mass as an intermediate step.

### 60.7 Numerical values and caveats

The quark masses in the particle data listings have been obtained by using a wide variety of methods. Each method involves its own set of approximations and uncertainties. In most cases, the errors are an estimate of the size of neglected higher-order corrections or other uncertainties. The expansion parameters for some of the approximations are not very small (for example, they are  $m_K^2/\Lambda_\chi^2 \sim 0.25$  for the SU(3) chiral expansion and  $\Lambda_{\text{QCD}}/m_b \sim 0.1$  for the heavy-quark expansion), so an unexpectedly large coefficient in a neglected higher-order term could significantly alter the results. Thus, before using a particular result, it is important to understand the possible limitations of the approach used to obtain it. It is also important to note that the quark mass values can be significantly different in the different schemes.

We have specified all masses in the  $\overline{\text{MS}}$  scheme. For light quarks, the renormalization scale has been chosen to be  $\mu = 2 \text{ GeV}$ . Quoting these masses at smaller values of  $\mu$ , where perturbative corrections become significantly larger, would introduce unnecessary uncertainties in the results. In fact, as lattice calculations, performed on finer and finer lattices, allow to determine quark masses, fully nonperturbatively, at larger and larger values of  $\mu$ , it may become advantageous to quote quark mass results at renormalization scales above 2 GeV, where perturbative uncertainties are smaller.

The heavy quark masses obtained using HQET, QCD sum rules, or lattice gauge theory are consistent with each other if they are all converted into the same scheme and scale. For these quarks it is conventional to choose the renormalization scale equal to the quark mass, so we have quoted  $\bar{m}_Q(\mu)$  at  $\mu = \bar{m}_Q$  for the  $c$  and  $b$  quarks. Given the small size of the charm quark mass, in the future it may become advantageous to quote its value at larger values of  $\mu$  so as not to introduce unnecessary perturbative uncertainties (see discussion above). Analyses of inclusive  $B$  meson decays have shown that other mass definitions lead to a better behaved perturbation series than for the  $\overline{\text{MS}}$  mass, and hence to more accurate mass values [67–70]. Thus, we have chosen to also give values for one of these, the  $b$  quark mass in the 1S scheme [67, 68]. Other schemes that

have been proposed are the PS-scheme [69], the kinetic scheme [71] and, most recently, the minimal renormalon-subtracted mass (MRS) [70] used in the lattice calculation [20].

If necessary, we have converted values in the original papers to our chosen scheme using two-loop formulæ. It is important to realize that our conversions introduce significant additional errors. In converting to the  $\overline{\text{MS}}$   $b$ -quark mass, for example, the three-loop conversions from the 1S and pole masses give values about 35 MeV and 135 MeV lower than the two-loop conversions. The uncertainty in  $\alpha_s(M_Z) = 0.1179 \pm 0.0010$  [1] gives an uncertainty of  $\pm 9$  MeV and  $\pm 21$  MeV respectively in the same conversions. We have not added these additional errors when we do our conversions. The  $\alpha_s$  value in the conversion is correlated with the  $\alpha_s$  value used in determining the quark mass, so the conversion error is not a simple additional error on the quark mass.

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